Endogenous debt constraints in collateralized economies with default penalties[*]

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Abstract

In infinite horizon economies with limited commitment, collateral requirements and default penalties, Páscoa and Seghir (2009) have shown that, in the absence of debt constraints, harsh default penalties may induce agents to run Ponzi schemes that jeopardize equilibrium existence. We appropriately modify the definition of finitely effective debt constraints, introduced by Levine and Zame (1996) (see also Levine and Zame (2002)), to encompass models with limited commitment, default penalties and collateral. Along this line, we introduce in the setting of Araujo, Páscoa and Torres-Martínez (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of actions with finite equivalent payoffs. We show that, independently of the level of default penalties, restricting plans to have finite equivalent payoffs rules out Ponzi schemes and guarantees the existence of an equilibrium that is compatible with the minimal ability to borrow and lend that we expect in our model. An interesting feature of our debt constraints is that they give rise to budget sets that coincide with the standard budget sets of economies having a collateral structure but no penalties (as defined in Araujo et al. (2002) and Kubler and Schmedders (2003)). This illustrates the hidden relation between finitely effective debt constraints and collateral requirements.

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1 Introduction

One of the main difficulties of extending sequential markets economies to an infinite horizon is associated with the existence of the so-called Ponzi schemes. In the absence of a terminal date agents would attempt to finance unbounded levels of consumption by renewing their credit at infinite. If such schemes are permitted, the agent's decision problem has no solution. Therefore, without debt constraints that limit the rate at which agents accumulate debt, equilibria fail to exist.

Broadly speaking three approaches have been proposed in the literature to deal with the specification of debt constraints in infinite horizon sequential markets models. The main difference among these lines of research hinges on the specific assumptions made about the enforcement of payments as well as the proposed default punishment.

The first approach, due to Magill and Quinzii (1994), Hernández and Santos (1996) and Levine and Zame (1996)(see also Levine and Zame (2002)), introduces debt constraints in economies where there is perfect enforcement and therefore no default (even on out of equilibrium paths). Magill and Quinzii (1994) argue in favor of implicit debt constraints that restrict budget sets to include portfolios whose value is a bounded sequence along the event tree. An interesting property of equilibria with implicit debt constraints is that it is always possible to find uniform bounds on short sales which are non-binding at those equilibria. Moreover, under reasonable assumptions on preferences, equilibria with implicit debt constraints coincide with equilibria with transversality type conditions that are often imposed in macroeconomic models (see Blanchard and Fisher (1989) and Ljungqvist and Sargent (2000)). Hernández and Santos (1996) argue in favor of debt constraints that impose a kind of solvency requirement. Households borrow against their current value of future endowment streams. When markets are incomplete, traders may not agree on current value prices. Hernández and Santos (1996) propose a special way of computing current value prices that takes into account the whole set of non-arbitrage price systems. Levine and Zame (1996) (see also Levine and Zame (2002)) offer an alternative formulation of the solvency requirement. They formalize borrowing constraints that restrict agents' debt to be repayable in finite time, that is, they impose debt constrains that are *finitely effective*. Stated differently, agents' actions are finitely effective when they are budget compatible with the threat that, at any period, agents may be restricted to have access to credit markets only for a finite number of periods. Finitely effective debt constraints provide a general characterization of debt constraints that are compatible with equilibrium. More precisely, Levine and Zame (1996) have shown that any loose and consistent debt constraints (see Levine and Zame (1996) for a precise definition) that rule out Ponzi schemes and ensure existence of an equilibrium reduce to be finitely effective. [1]

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[1] See also Hernández and Santos (1996) for a similar discussion.

The second approach, due to Kehoe and Levine (1993) (see also Kehoe and Levine (2001)), Zhang (1997) and Alvarez and Jermann (2000), explores debt constraints in economies where commitment is limited and there is a severe punishment for default: if agents do not honor their debts, they are excluded from participating in the asset markets in future periods. In such a setting the authors argue for self-enforcing constraints (so-called participation constraints) that are tight enough to prevent default at equilibrium but simultaneously are loose enough to allow for as much risk sharing as possible.

The third and most recent approach to deal with Ponzi schemes also considers models with limited commitment. However, contrary to self-enforcing borrowing constraints (à la Alvarez and Jermann (2000)) that prevent default at equilibrium, this research line addresses the issue of Ponzi schemes in economies where default may appear at equilibrium. It is motivated by the empirical observation that modern economies experience a substantial amount of default and bankruptcy.[2] One of the most important and widespread means of securing loans and lowering the level of default in financial markets is collateral.[3] Araujo et al. (2002) (see also Kubler and Schmedders (2003)) showed that, without imposing any debt constraints or transversality conditions, Ponzi schemes are ruled out in economies where collateral is the only mechanism that enforces agents to (partially) pay their debts. The intuition behind their result is as follows. Combining short-sales with the purchase of collateral constitutes a joint operation that yields non-negative returns. By non-arbitrage, at equilibrium, the price of the collateral exceeds the price of the asset, implying that collateral costs exceed the value of loans. Therefore, it becomes impossible to pay a previous debt by issuing new debt.

In most economic systems collateral is not the only mean of securing loans. The default option usually entails additional economic consequences.[4] A possible reason

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[2] Nowadays, there is a vast literature on default that dates back to the seminal contributions of Shubik (1972), Shubik and Wilson (1977) and Dubey and Shubik (1979). Default was introduced in a general equilibrium setting by Dubey, Geanakoplos and Shubik (1990) and Zame (1993). Modern theoretical contributions on default include among others, Dubey, Geanakoplos and Zame (1995), Geanakoplos (1997), Geanakoplos and Zame (2002), Araujo et al. (2002), Kubler and Schmedders (2003), Dubey, Geanakoplos and Shubik (2005), Fostel and Geanakoplos (2008), Páscoa and Seghir (2009), Revil and Torres-Martínez (2010). There are also important contributions on default, collateral and credit constraints in macroeconomics (see Bernanke, Gertler and Gilchrist (1996), Kiyotaki and Moore (1997) and Caballero and Krishnamurthy (2001)). This literature emphasizes the feedback from the fall in collateral prices to a fall in borrowing capacity. Recently, Chatterjee, Corbae, Nakajima and Ríos-Rull (2007) and Livshits, MacGee and Tertilt (2007) have calibrated macroeconomic models with incomplete markets and default and used them to address various policy issues.

[3] Collateral-using activities have expanded rapidly in recent years. Financial institutions extensively employ collateral in lending, in securities trading and derivative markets and in payment and settlement systems. Central banks generally require collateral in their credit operations. Common examples of collateralized lending are home mortgages, margin purchases of securities, overnight repurchase agreements and pawn shop loans.

[4] For instance, if an agent files for bankruptcy under Chapter 7 of the U.S. bankruptcy code, the following things may happen (see Chatterjee et al. (2007)): (1) he is not allowed to save and his existing savings will be completely garnished; (2) he has to pay a proportion of the current income as cost of filling for bankruptcy; (3) a proportion of his current labor income is garnished; (4) his credit history turns bad

is that the effectiveness of collateral is rather limited in the presence of large negative shocks in the value of collateral guarantees. One approach to model additional enforcement mechanisms is to introduce linear utility penalties (see Zame (1993), Dubey et al. (2005) and the literature cited therein). Following Zame (1993) and Dubey et al. (2005), these penalties might be interpreted as the consequences (directly assessed in terms of utility) of some third party punishment such as prison terms and pangs of conscience, and/or of some non-modeled economic punishment such as exclusion from credit markets and garnishing of future income.

A surprising result found by Páscoa and Seghir (2009) is that the introduction of default penalties in the model of Araujo et al. (2002) may induce payments besides the value of the collateral and lead to the reappearance of Ponzi schemes. The intuition is as follows. When penalties are severe, agents have incentives to repay more than the value of the depreciated collateral. In this case, the joint operation of combining short sales with the purchase of collateral no longer yields non-negative returns. Therefore, loans may exceed collateral costs and agents may run Ponzi schemes.

One may think that the reappearance of Ponzi schemes is related to the particular additional enforcement mechanism (linear utility penalties) Páscoa and Seghir (2009) have considered. However, Revil and Torres-Martínez (2010) showed that any effective additional enforcement mechanism implies the non-existence of physically feasible optimal plans.[5] That is, any effective additional enforcement mechanism gives rise to Ponzi schemes in infinite horizon collateralized economies. Hence, it is the effectiveness of the mechanism that induces agents to run a Ponzi scheme, not the mechanism per se.

Páscoa and Seghir (2009) claimed that collateral still avoids Ponzi schemes provided that default penalties are moderate, in the sense that the penalty associated with the maximal default for a physically feasible plan is less than the utility from consuming the current endowment. Their claim appears to be intuitive. If default penalties are moderate, then default does not hurt much. Therefore, moderate default penalties should not induce payments besides the value of the collateral, that is, they should not be effective. Although intuitive, this result is not correct. As shown by Martins-da-Rocha and Vailakis (2010), even moderate default penalties are not always compatible with the existence of equilibrium. However, it is possible to find a condition relating default penalties with the primitives of the model that captures the intuition behind the effectiveness of default penalties conjectured by Páscoa and Seghir (2009).[6]

This paper follows another route and argues in favor of implicit debt constraints. We believe that this direction is more relevant from an economic perspective for two reasons. First, to the extent that utility penalties are proposed as a convenient shortcut for assessing the consequences of non-modeled enforcement mechanisms, it is difficult

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and he is excluded from the loan market.

[5] An enforcement mechanism is said effective if it entails payments besides the value of the collateral at all nodes of a subtree.

[6] Martins-da-Rocha and Vailakis (2010) proved that equilibrium existence is restored if the marginal utility of consuming the collateral is eventually larger than the marginal penalty agents suffer from defaulting on their promises.

to evaluate whether the condition proposed by Martins-da-Rocha and Vailakis (2010) (or any other condition relating default penalties to primitives of the model) is economically meaningful. Second, the condition on penalties proposed by Martins-da-Rocha and Vailakis (2010) is only biting in models where the collateral takes the form of a particular bundle of goods that, when consumed, increases agents' welfare. However, in many models (see Kubler and Schmedders (2003) and Kiyotaki and Moore (1997)) proposed in the literature the collateral requirements take the form of a productive asset that cannot be consumed directly by the agents. In those settings, it becomes more difficult to come up with a condition on penalties that rules out Ponzi schemes.

We explore debt constraints that are compatible with equilibrium (and in particular excluding Ponzi schemes) and simultaneously allow for as much risk sharing as possible in infinite horizon collateralized economies with default penalties. We argue in favor of endogenous and finitely effective debt constraints, similar to those proposed by Levine and Zame (1996). In environments with limited commitment, this kind of borrowing constraints has not attracted much attention so far. A direct adaptation of finitely effective debt constraints à la Levine and Zame (1996) in those environments does not help to control debt along time. The reason is that when commitment is limited, an agent can always satisfy his budget restrictions having access to financial markets for a finite number of periods. He can do this by simply defaulting on his promises. Therefore, requiring finite-time solvency à la Levine and Zame (1996) does not restrict budget sets. In particular, it does not exclude Ponzi schemes. The paper addresses this issue by modifying appropriately the definition of finitely effective debt constraints to encompass economies with limited commitment and (possible) default at equilibrium. Working in this direction, we impose debt constraints by introducing in the setting of Araujo et al. (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of actions with *finite equivalent payoffs*. We argue that the proposed debt constraints provide a meaningful formulation of the solvency requirement in those models.

The implementation of those constraints in decentralized markets can be given the following interpretation. At any time period each agent conceives that he may be restricted (by an agency or by "the market") to have access to credit markets for a finite number of periods. The presence of such a threat makes agents adjust their trading strategies in such a way that, if they only have access to credit markets for a finite number of periods, they can modify their consumption and investment plans so that the the loss of utility, compared to what was initially planned, is minimal. The agency does not need to know agents' characteristics. The mere fact that agents believe that the agency has the legal authority to exclude them from participating in credit markets at some point rules out Ponzi schemes and leads to an equilibrium. In that respect, our interpretation is partly objective (market based) since it requires the presence of an agency, and partly subjective (self-monitoring) since it is the agents who restrict their trading strategies.

An interesting observation is that there is a close relation between our proposed budget sets and the budget sets of Levine and Zame (1996) as well as the budget sets defined through collateral obligations and no additional punishments (Araujo et al. (2002) and Kubler and Schmedders (2003)). When there is full commitment (and payments are fully enforced) our concept of plans with finitely equivalent payoffs coincides with the concept of plans with finitely effective debt introduced by Levine and Zame (1996). Most important, we show that the budget feasible plans in economies with a collateral structure and no default penalties have finite equivalent payoffs and vice versa. In other words, when there are collateral requirements but no default penalties, our budget set coincides with the standard one defined in Araujo et al. (2002) and Kubler and Schmedders (2003). This equivalence is valid for any price process and illustrates the hidden relation between finitely effective debt constraints and collateral requirements. Our approach to debt constraints is certainly not the only one possible. Instead of adapting the restrictions proposed by Levine and Zame (1996), we may have followed another route by considering those proposed by Magill and Quinzii (1994) or Hernández and Santos (1996). However, the equivalence between our budget set and those in models with collateral requirements and no default penalties shows that our approach to control debt is much more suitable for models with limited commitment and collateral requirements.

The paper is structured as follows. In Section 2 we set out the model, introduce notation and the equilibrium concept in the absence of borrowing constraints. Section 3 contains the assumptions imposed on the characteristics of the economy. In Section 4 we present and discuss the new debt constraints we impose on budget feasible plans. We also introduce an equilibrium concept associated with those constraints and highlight its relation with the equilibrium concepts introduced by Levine and Zame (1996) and Araujo et al. (2002). Section 5 proves the existence of what we term *equilibrium with finite equivalent payoffs* under a mild condition on default penalties. Section A contains some technical results.

2 The Model

The model is essentially the one developed in Araujo et al. (2002) and extended by Páscoa and Seghir (2009) to allow for the possibility of linear default penalties. It can also be seen as an infinite horizon extension of the model proposed by Dubey et al. (2005).

2.1 Uncertainty and time

Let $\mathscr{T} \equiv \{0, 1, \dots, t, \dots\}$ denote the set of time periods and let S be a (infinite) set of states of nature. The available information at period $t \in \mathscr{T}$ is the same for each agent and is described by a finite partition P_t of S. Information is revealed along time, i.e., the partition P_{t+1} is finer than P_t for every t. Every pair (t, σ) where σ is a set in P_t is called a node. The set of all nodes is denoted by D and is called the event tree. We assume that there is no information at $t = 0$ and we denote by $\xi_0 = (0, S)$ the initial node. If $\xi = (t, \sigma)$ belongs to the event tree, then t is denoted by $t(\xi)$. We say that

$\xi' = (t', \sigma')$ is a successor of $\xi = (t, \sigma)$ if $t' \geqslant t$ and $\sigma' \subset \sigma$; we use the notation $\xi' \geqslant \xi$. We denote by ξ^+ the set of immediate successors defined by

$$\xi^+ \equiv \{\xi' \in D \ : \ t(\xi') = t(\xi) + 1\}.$$

Because P_t is finer than P_{t-1} for every $t > 0$, for a given node $\xi \neq \xi_0$, there is a unique node ξ^- in D such that ξ is an immediate successor of ξ^-. Given a period $t \in \mathscr{T}$ we denote by D_t the set of nodes at period t, i.e., $D_t \equiv \{\xi \in D \ : \ t(\xi) = t\}$. The set of nodes up to period t is denoted by D^t, i.e., $D^t \equiv \{\xi \in D \ : \ t(\xi) \leqslant t\}$.

2.2 Agents and commodities

There exists a finite set L of commodities available for trade at every node $\xi \in D$. We interpret $x(\xi) \in \mathbb{R}^L_+$ as a claim to consumption at node ξ. We also write $\mathbf{1}_{\{\ell\}} \in \mathbb{R}^L_+$ for the commodity bundle consisting of one unit of commodity $\ell \in L$ and nothing else. We depart from the usual intertemporal models by allowing for some commodities to be *non-perishable*, that is, we allow for storable and durable goods as well as for commodities that may serve as physical assets (i.e., Lucas trees). Transformation of commodities is represented by a family $(Y(\xi))_{\xi \in D}$ of linear functionals $Y(\xi)$ from $\mathbb{R}^L_+$ to $\mathbb{R}^L_+$. The bundle $Y(\xi)z(\xi^-)$ represents what is obtained at node ξ if the bundle $z(\xi^-) \in \mathbb{R}^L_+$ is purchased at node ξ^-. We say that the commodity ℓ is perishable at node ξ^- if $Y(\xi)\mathbf{1}_{\{\ell\}}$ is the zero vector in $\mathbb{R}^L_+$, and non-perishable otherwise. At each node there are spot markets for trading every commodity. We let $p = (p(\xi))_{\xi \in D}$ be the spot price process where $p(\xi) = (p(\xi, \ell))_{\ell \in L} \in \mathbb{R}^L_+$ is the price vector at node ξ.

There is a finite set I of infinitely lived agents. Each agent $i \in I$ is characterized by an endowment process $\omega^i = (\omega^i(\xi))_{\xi \in D}$ where $\omega^i(\xi) = (\omega^i(\xi, \ell))_{\ell \in L} \in \mathbb{R}^L_+$ denotes the endowment available at node ξ. Each agent chooses a consumption process $x = (x(\xi))_{\xi \in D}$ where $x(\xi) \in \mathbb{R}^L_+$. We denote by X the set of consumption processes. The utility function $U^i : X \longrightarrow [0, +\infty]$ is assumed to be additively separable, i.e.,

$$U^i(x) = \sum_{\xi \in D} u^i(\xi, x(\xi))$$

where $u^i(\xi, \cdot) : \mathbb{R}_+ \longrightarrow [0, \infty)$.

2.3 Assets and collateral

There is a finite set J of short-lived real financial assets available for trade at each node. For each asset j, the bundle yielded at node ξ is denoted by $A(\xi, j) \in \mathbb{R}^L_+$. We let $q = (q(\xi))_{\xi \in D}$ be the asset price process where $q(\xi) = (q(\xi, j))_{j \in J} \in \mathbb{R}^J_+$ represents the asset price vector at node ξ. We denote by $\theta^i(\xi) \in \mathbb{R}^J_+$ the vector of purchases and by $\varphi^i(\xi) \in \mathbb{R}^J_+$ the vector of short-sales at each node ξ.

Following Araujo et al. (2002) and Páscoa and Seghir (2009) (see also Geanakoplos (1997) and Geanakoplos and Zame (2002)), assets are collateralized in the sense that for every unit of asset j sold at a node ξ, agents should buy a collateral bundle $C(\xi, j) \in$

$\mathbb{R}^L_+$ that protects lenders in case of default. We assume that payments can be enforced only through the seizure of the collateral. At a node ξ, agent i should deliver the promise $V(p,\xi)\varphi^i(\xi^-)$ where

$$V(p,\xi) = (V(p,\xi,j))_{j\in J} \quad \text{and} \quad V(p,\xi,j) \equiv p(\xi)A(\xi,j).$$

However, agent i may decide to default and choose a delivery $d^i(\xi,j)$ in units of account. Since the collateral can be seized, this delivery must satisfy

$$d^i(\xi,j) \geqslant D(p,\xi,j)\varphi^i(\xi^-,j)$$

where

$$D(p,\xi,j) \equiv \min\{p(\xi)A(\xi,j), p(\xi)Y(\xi)C(\xi^-,j)\}.$$

Remark 2.1. Kubler and Schmedders (2003) propose a model where the collateral requirements are imposed in terms of physical assets. We show that their model is a particular case of the model proposed by Araujo et al. (2002). In that respect whenever we are referring to the model proposed by Araujo et al. (2002) we are also referring to the one proposed by Kubler and Schmedders (2003).

If there is a specific commodity $g \in L$ satisfying the following properties, then this commodity can be interpreted as a physical asset or a Lucas tree.

(i) At initial node ξ_0, each agent i has an initial endowment $\omega^i(\xi_0,g) \geqslant 0$ of commodity g which represents his share of the tree. At subsequent nodes $\xi > \xi_0$, agent i has no initial endowment in commodity g.

(ii) One unit of commodity g purchased at node ξ delivers at node $\mu \in \xi^+$ the bundle

$$y(\mu) \equiv Y(\mu)\mathbf{1}_{\{g\}} \in \mathbb{R}^L_+.$$

The g-th coordinate $y(\mu,g)$ is equal to 1, i.e., the physical asset is long lived.

(iii) Each agent i is indifferent with respect to commodity g, i.e., for each agent $i \in I$, for each node $\xi \in D$, for each consumption bundle $c \in \mathbb{R}^L_+$, we have

$$u^i(\xi, c + \mathbf{1}_{\{g\}}) = u^i(\xi, c).$$

(iv) In every successor node $\mu \in \xi^+$, the transformed bundle of one unit of commodity g purchased at any node ξ, is a desirable bundle, i.e., $y(\mu)$ is a bundle in $\mathbb{R}^L_+$ such that for each consumption bundle $c \in \mathbb{R}^L_+$, we have[7]

$$u^i(\mu, c + y(\mu)) > u^i(\mu, c).$$

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[7] Since each agent i is indifferent with respect to commodity g, the bundle delivered by the tree must satisfy $y(\mu,\ell) > 0$ for at least one commodity $\ell \neq g$.

If at every node $\xi \in D$, the collateral bundle $C(\xi, j)$ is only in terms of commodity g, then the collateral structure of our model (and the one in Araujo et al. (2002) and Páscoa and Seghir (2009)) reduces to the one considered by Kubler and Schmedders (2003).

Following Dubey et al. (2005) and Páscoa and Seghir (2009), we assume that agent i feels a disutility $\lambda^i(\xi, j) \in [0, +\infty]$ from defaulting. More precisely, if an agent defaults at node ξ, then he suffers at $t = 0$, the disutility

$$\sum_{j \in J} \lambda^i(\xi, j) \frac{\left[V(p, \xi, j)\varphi^i(\xi^-, j) - d^i(\xi, j)\right]^+}{p(\xi)\nu(\xi)}$$

where $(\nu(\xi))_{\xi \in D}$ is an exogenously specified process in $\mathbb{R}^L_{++}$ that is uniformly bounded away from 0.[8] In that case, agent i may have an incentive to deliver more than the minimum between his debt and the depreciated value of his collateral, i.e., we may have $d^i(\xi, j) > D(p, \xi, j)\varphi^i(\xi^-, j)$.

As in Dubey et al. (2005) assets are thought as pools. At each node ξ the sales $\varphi^i(\xi, j)$ are pooled at the market for asset j. The deliveries $d^i(\xi, j)$ on asset j are also pooled and the buyers of pool j receive a pro rata share of all its different sellers' deliveries. Each share of pool j delivers the fraction $V(\kappa, p, \xi, j)$ of its promise $V(p, \xi, j)$ defined by

$$V(\kappa, p, \xi, j) = \kappa(\xi, j)V(p, \xi, j) + (1 - \kappa(\xi, j))D(p, \xi, j)$$

where $\kappa(\xi, j) \in [0, 1]$.[9] The buyer of asset j does not need to know the identities of the sellers or the quantities of their sales. All that matters to him is the price $q(\xi, j)$ of one unit of asset and the anticipated delivery rates $(\kappa(\mu, j))_{\mu \in \xi^+}$.

2.4 Budget constraints

We let A be the space of adapted processes $a = (a(\xi))_{\xi \in D}$ with

$$a(\xi) = (x(\xi), \theta(\xi), \varphi(\xi), d(\xi))$$

where

$$x(\xi) \in \mathbb{R}^L_+, \quad \theta(\xi) \in \mathbb{R}^J_+, \quad \varphi(\xi) \in \mathbb{R}^J_+, \quad d(\xi) \in \mathbb{R}^J_+$$

and by convention

$$a(\xi_0^-) = (x(\xi_0^-), \theta(\xi_0^-), \varphi(\xi_0^-), d(\xi_0^-)) = (0, 0, 0, 0).$$

In each decision node $\xi \in D$, agent i's choice $a^i = (x^i, \theta^i, \varphi^i, d^i) \in A$ must satisfy the following constraints:

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[8] More precisely, we assume that there exists $\underline{\nu} > 0$ such that for every node $\xi \in D$ and every commodity $\ell \in L$, we have $\nu(\xi, \ell) \geqslant \underline{\nu}$.

[9] If all the sellers of asset j fully deliver on their promises then $\kappa(\xi, j) = 1$, while if all sellers fully default on their promises then $\kappa(\xi, j) = 0$.

(a) solvency constraint:

$$p(\xi)x^i(\xi)+\sum_{j\in J}d^i(\xi,j)+q(\xi)\theta^i(\xi)$$
$$\leqslant p(\xi)[\omega^i(\xi)+Y(\xi)x^i(\xi^-)]+V(\kappa,p,\xi)\theta^i(\xi^-)+q(\xi)\varphi^i(\xi); \tag{2.1}$$

(b) collateral requirement:

$$C(\xi)\varphi^i(\xi)\leqslant x^i(\xi); \tag{2.2}$$

(c) minimum delivery:

$$\forall j\in J,\quad D(p,\xi,j)\varphi^i(\xi^-,j)\leqslant d^i(\xi,j). \tag{2.3}$$

2.5 The payoff function

Assume that $\pi=(p,q,\kappa)$ is a process of prices and delivery rates. Consider that agent i has chosen the plan $a=(x,\theta,\varphi,d)\in A$. He gets the utility $U^i(x)\in[0,\infty]$ defined by

$$U^i(x)=\sum_{\xi\in D}u^i(\xi,x(\xi))$$

but he suffers the disutility $W^i(p,a)\in[0,\infty]$ defined by

$$W^i(p,a)=\sum_{\xi>\xi_0}\sum_{j\in J}\lambda^i(\xi,j)\frac{[V(p,\xi,j)\varphi(\xi^-,j)-d(\xi,j)]^+}{p(\xi)\nu(\xi)}.$$

We would like to define the payoff $\Pi^i(p,a)$ of the plan a as the following difference

$$\Pi^i(p,a)=U^i(x)-W^i(p,a).$$

Unfortunately, $\Pi^i(p,a)$ may not be well defined if both $U^i(x)$ and $W^i(p,a)$ are infinite.[10] We propose to consider the binary relation $\succ_{i,p}$ defined on A by

$$\widetilde{a}\succ_{i,p}a\Longleftrightarrow\exists\varepsilon>0,\quad\exists T\in\mathbb{N},\quad\forall t\geqslant T,\quad\Pi^{i,t}(p,\widetilde{a})\geqslant\Pi^{i,t}(p,a)+\varepsilon$$

where

$$\Pi^{i,t}(p,a)\equiv U^{i,t}(x)-W^{i,t}(p,a),\quad U^{i,t}(x)\equiv\sum_{\xi\in D^t}u^i(\xi,x(\xi))$$

and

$$W^{i,t}(p,a)\equiv\sum_{\xi\in D^t\setminus\{\xi_0\}}\sum_{j\in J}\lambda^i(\xi,j)\frac{[V(p,\xi,j)\varphi(\xi^-,j)-d(\xi,j)]^+}{p(\xi)\nu(\xi)}.$$

Observe that if $\Pi^i(p,\widetilde{a})$ and $\Pi^i(p,a)$ are finite then

$$\widetilde{a}\succ_{i,p}a\Longleftrightarrow\Pi^i(p,\widetilde{a})>\Pi^i(p,a).$$

The set $\mathrm{Pref}^i(p,a)$ of plans strictly preferred to plan a by agent i is defined by

$$\mathrm{Pref}^i(p,a)\equiv\{\widetilde{a}\in A:\ \widetilde{a}\succ_{i,p}a\}.$$

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[10] This issue is ignored by Páscoa and Seghir (2009).

2.6 The equilibrium concept

We denote by Ξ the set of prices and delivery rates (p,q,κ) satisfying

$$\forall \xi \in D, \quad p(\xi) \in \mathbb{R}^L_{++}, \quad q(\xi) \in \mathbb{R}^J_{+}, \quad \kappa(\xi) \in [0,1]^J \tag{2.4}$$

and

$$\sum_{\ell \in L} p(\xi,\ell) + \sum_{j \in J} q(\xi,j) = 1.$$

We denote by $\mathrm{cl}\,\Xi$ the closure of Ξ under the weak topology.[11]

Given a process (p,q,κ) of commodity prices, asset prices and delivery rates, we denote by $B^i(p,q,\kappa)$ the set of plans $a = (x,\theta,\varphi,d) \in A$ satisfying constraints (2.1), (2.2) and (2.3). The demand $d^i(p,q,\kappa)$ is defined by

$$d^i(p,q,\kappa) \equiv \{a \in B^i(p,q,\kappa) \ : \ \mathrm{Pref}^i(p,a) \cap B^i(p,q,\kappa) = \emptyset\}.$$

Definition 2.1. *A competitive equilibrium for the economy $\mathscr{E}$ is a family of prices and delivery rates $(p,q,\kappa) \in \Xi$ and an allocation $\boldsymbol{a} = (a^i)_{i\in I}$ with $a^i \in A$ such that*

(a) *for every agent i, the plan a^i is optimal, i.e.,*

$$a^i \in d^i(p,q,\kappa);$$

(b) *commodity markets clear at every node, i.e.,*

$$\sum_{i\in I} x^i(\xi_0) = \sum_{i \in I} \omega^i(\xi_0) \tag{2.5}$$

and for all $\xi \neq \xi_0$,

$$\sum_{i\in I} x^i(\xi) = \sum_{i\in I} \left[\omega^i(\xi) + Y(\xi)x^i(\xi^-)\right]; \tag{2.6}$$

(c) *asset markets clear at every node, i.e., for all $\xi \in D$,*

$$\sum_{i\in I} \theta^i(\xi) = \sum_{i\in I} \varphi^i(\xi); \tag{2.7}$$

(d) *deliveries match at every node, i.e., for all $\xi \neq \xi_0$ and all $j \in J$,*

$$\sum_{i\in I} V(\kappa,p,\xi,j)\theta^i(\xi^-,j) = \sum_{i\in I} d^i(\xi,j). \tag{2.8}$$

The set of allocations $\boldsymbol{a} = (a^i)_{i\in I}$ in A satisfying the market clearing conditions (2.5), (2.6) and (2.7) is denoted by F. Each allocation in F is called physically feasible. A plan $a^i \in A$ is called physically feasible if there exists a physically feasible allocation $\boldsymbol{b}$ such that $a^i = b^i$. The set of physically feasible plans is denoted by F^i. We denote by $\mathrm{Eq}(\mathscr{E})$ the set of competitive equilibria for the economy $\mathscr{E}$.

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[11] The process (p,q,κ) belongs to $\mathrm{cl}\,\Xi$ if the condition "$p(\xi) \in \mathbb{R}^L_{++}$" in (2.4) is replaced by "$p(\xi) \in \mathbb{R}^L_{+}$".

3 Assumptions

For each agent i, we denote by $\Omega^i = (\Omega^i(\xi))_{\xi \in D}$ the process of accumulated endowments, defined recursively by

$$\Omega^i(\xi_0) = \omega^i(\xi_0) \quad \text{and} \quad \forall \xi > \xi_0, \quad \Omega^i(\xi) = Y(\xi)\Omega^i(\xi^-) + \omega^i(\xi).$$

The process $\sum_{i \in I} \Omega^i$ of accumulated aggregate endowments is denoted by Ω. This section describes the assumptions imposed on the characteristics of the economy. It should be clear that these assumptions always hold throughout the paper.

Assumption 3.1 (Agents)**.** For every agent i,

(H.1) the process of accumulated endowments is strictly positive and uniformly bounded from above, i.e.,

$$\exists \overline{\Omega}^i \in \mathbb{R}^L_{++}, \quad \forall \xi \in D, \quad \Omega^i(\xi) \in \mathbb{R}^L_{++} \quad \text{and} \quad \Omega^i(\xi) \leqslant \overline{\Omega}^i;$$

(H.2) for every node ξ, the utility function $u^i(\xi, \cdot)$ is concave, continuous and strictly increasing,[12] with $u^i(\xi, 0) = 0$;

(H.3) the infinite sum $U^i(\Omega)$ is finite.

Assumption 3.2 (Financial assets)**.** For every asset j and node ξ, the collateral $C(\xi, j)$ is not zero.

Remark 3.1. Assumptions 3.1 and 3.2 are classical in the literature of infinite horizon models with collateral requirements (see e.g., Araujo et al. (2002) and Páscoa and Seghir (2009)).

4 Debt constraints

In this section, we show how to adapt the *finitely effective* debt constraints proposed by Levine and Zame (1996) to infinite horizon models with limited commitment and default penalties. While keeping the minimal ability to borrow and lend that we expect in our model, we prove that the proposed constraints are compatible with equilibrium (precluding agents to run Ponzi schemes). Moreover, our constraints appear to have an additional appealing feature. We prove that the budget sets associated with those constraints coincide with the standard budget sets of economies having a collateral structure but no penalties (as defined in Araujo et al. (2002) and Kubler and Schmedders (2003)).

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[12] We impose that the function $u^i(\xi, \cdot)$ is strictly increasing which is not compatible with the interpretation of a commodity as a Lucas tree. This assumption was made for expositional purposes and can be weakened as follows: for every ξ the function $u^i(\xi, \cdot)$ is non-decreasing and there exists a commodity ℓ that is strictly desirable in the sense that for every pair x, y in $\mathbb{R}^L_+$, we have $u^i(\xi, x+y) > u^i(\xi, x)$ provided that $y(\ell) > 0$.

4.1 Infinite default penalties

When default penalties are infinite and the collateral requirements are zero, our model reduces to the one studied by Magill and Quinzii (1994) and Levine and Zame (1996). In the absence of debt constraints, an equilibrium may not exist: all traders would attempt to finance unbounded levels of consumption by unbounded levels of borrowing. To rule out Ponzi schemes, Levine and Zame (1996) (see also Levine and Zame (2002)) formalize the concept of plans with *finitely effective* debt by requiring agents' actions to be budget compatible with the threat that, at any period, agents may be restricted to have access to credit markets for a finite number of periods. Equivalently, an agent's debt is finitely effective if at any period, the debt is repayable within a finite horizon. More formally, we consider the following definition due to Levine and Zame (1996).

Definition 4.1. *A plan $a \in B^i(p,q,\kappa)$ is said to have finitely effective debt, if for each period $t \geqslant 0$, there exists a period $T > t$ and a plan $\widehat{a}$ also in the budget set $B^i(p,q,\kappa)$ such that*

(i) *up to period t both plans coincide, i.e.,*

$$\forall \xi \in D^t, \quad \widehat{a}(\xi) = a(\xi);$$

(ii) *at every node after period T, there is solvency without new loans, i.e.,*

$$\forall \xi \in D, \quad t(\xi) \geqslant T \Longrightarrow \widehat{\varphi}(\xi) = 0.$$

The intuition behind Definition 4.1 can be better understood if we think about the role of those restrictions in the finite horizon framework. No short selling at the terminal date implicitly imposes a solvency requirement at earlier dates. That is, at any node agents should hold an amount of debt that they will be able to repay by the end of the terminal date. In the absence of a terminal date, it is necessary to impose explicitly or implicitly that solvency requirement.

Remark 4.1. Consider the following notation. For each period t, we denote by A^t the set of plans $a \in A$ where $a(\xi) = (0,0,0,0)$ for each ξ such that $t(\xi) > t$. If a is a plan in A and t is a period, we denote by $a\mathbf{1}_{[0,t]}$ the plan in A^t which coincides with a for every node $\xi \in D^t$. Following this notation, a plan a has a finitely effective debt if for each period $t \geqslant 0$, there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}$$

where C^T is the set of plans a in A satisfying

$$\forall \xi \in D, \quad t(\xi) \geqslant T \Longrightarrow \widehat{\varphi}(\xi) = 0.$$

The following proposition provides an equivalent characterization of plans with finitely effective debt. This alternative characterization will be proven particularly useful in the process of modifying finitely effective constraints to encompass models with limited commitment.

Proposition 4.1. Assume that the default penalty is infinite and consider a budget feasible plan $a \in B^i(p,q,\kappa)$ with a finite utility $U^i(x) < \infty$. The plan a has a finitely effective debt, if and only if, it has *finite equivalent utility* in the sense that for every period $t \geqslant 0$ and every $\varepsilon > 0$ there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T, \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]} \quad \text{and} \quad \inf_{\tau \geqslant T} \left[U^{i,\tau}(p,\widehat{a}) - U^{i,\tau}(p,a) \right] \geqslant -\varepsilon.$$

Proof of Proposition 4.1. Let $a \in B^i(p,q,\kappa)$ be a budget feasible plan with a finite utility $U^i(x) < \infty$. It is obvious that if a has finite equivalent utility, then it has a finitely effective debt. The converse deserves more attention. Assume that the plan a has a finitely effective debt. Fix a period t and $\varepsilon > 0$. If we apply the definition to the period t, we get the existence of a period $T > t$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]}.$$

Unfortunately, we do not know if $U^{i,T}(\widehat{x}) \geqslant U^{i,T}(x) - \varepsilon$. However, we know that the utility $U^i(x)$ is finite. Therefore, there exists $t' > t$ such that

$$\sum_{s > t'} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \leqslant \varepsilon. \tag{4.1}$$

Now, applying the definition of *finitely effective debt* for the period t', there exists a period $T > t'$ and a plan $\widehat{a}$ such that

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad a\mathbf{1}_{[0,t']} = \widehat{a}\mathbf{1}_{[0,t']}.$$

Now fix $\tau \geqslant T$. Since $T > t'$, we have

$$U^{i,\tau}(\widehat{x}) \geqslant U^{i,T}(\widehat{x}) \geqslant U^{i,t'}(\widehat{x}) = U^{i,t'}(x) \geqslant U^{i,\tau}(x) - \sum_{t' < s \leqslant \tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)).$$

It follows from (4.1) that $U^{i,\tau}(\widehat{x}) \geqslant U^{i,\tau}(x) - \varepsilon$. □

4.2 Finite default penalties

The concept of finitely effective debt constraints makes perfect sense in models with full enforcement and perfect commitment (i.e., no default). However, with limited commitment, imposing finitely effective debt constraints does not help to control debt along time. We provide an explanation below. Let $a = (x, \theta, \varphi, d)$ be a plan in $B^i(p,q,\kappa)$ and t be any period. Consider the plan $\widehat{a}$ defined by

$$\forall \xi \in D, \quad \widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) \leqslant t \\ (\omega^i(\xi), 0, 0, D(p,\xi)\varphi(\xi^-)) & \text{if} \quad t(\xi) = t+1 \\ (\omega^i(\xi), 0, 0, 0) & \text{if} \quad t(\xi) > t+1. \end{cases}$$

This plan belongs to the set $B^i(p,q,\kappa) \cap C^{t+1}$ and coincides with a on every node up to period t. That is, under limited commitment, any plan $a \in B^i(p,q,\kappa)$ has finitely effective debt according to Definition 4.1. Agents can always default up to the minimum value between their debt and the depreciated value of their collateral. Therefore, there is no hope to bound debt along time.

We propose hereafter to modify the definition of endogenous debt constraints to encompass models with limited commitment and finite default penalties. The point of our departure is Proposition 4.1 where it is shown that, when default penalties are infinite, restricting plans to have finitely effective debt is equivalent to restricting plans to have finite equivalent utility. This equivalence breaks down in the presence of finite default penalties. In this case, we proceed by replacing "utility" by "payoff" and we introduce the concept of plans with *finite equivalent payoffs*. We claim that requiring plans to have finite equivalent payoffs provides an appropriate adaptation of finitely effective debt constraints to models with limited commitment and finite default penalties. The formal definition is as follows.

Definition 4.2. *A plan a in the budget set $B^i(p,q,\kappa)$ has finite equivalent payoffs if for every period $t \geqslant 0$ and every $\varepsilon > 0$ there exists a subsequent period $T > t$ and a plan $\widehat{a}$ such that*

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T, \quad a\mathbf{1}_{[0,t]} = \widehat{a}\mathbf{1}_{[0,t]} \quad \text{and} \quad \inf_{\tau \geqslant T} \left[\Pi^{i,\tau}(p,\widehat{a}) - \Pi^{i,\tau}(p,a) \right] \geqslant -\varepsilon.$$

The introduction of debt constraints raises issues related to the implementation of those constraints in decentralized markets. We propose hereafter an interpretation of our proposed debt constraints that is partly objective (market based) since it requires the presence of an agency, and partly subjective (self-monitoring) since it is the agents who restrict their trading strategies. When making a plan a^i, agent i conceives that, at any period t, there is a possibility that an agency will not allow him to have access to credit markets forever. The agent may believe that he will be able to negotiate with the agency on the number of periods he will still continue having access to credit markets. In this case, the agent will restrict himself on choosing plans such that, in the bad situation where the agency restricts his access to credit markets, he will not loose too much payoff. That is, for any period t, he should be able to deviate from the chosen plan a^i choosing another budget feasible plan $\widehat{a}^i$, different from a^i only after period t, satisfying the following properties: there is participation in credit markets only for a finite number of periods, and the payoff associated to $\widehat{a}^i$ is close to the payoff he would have enjoyed with the initial plan a^i. One could alternatively argue for a pure market based interpretation where the central authority enforces the borrowing limits at any node. We think that our interpretation is more appealing from an economic point of view. This is because it only requires the central authority to have the legal power to exclude an agent from credit markets. A pure market based interpretation requires a central authority that knows the agents' characteristics.[13]

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[13] An objective or subjective interpretation of participation constraints à la Kehoe and Levine (1993) (see

We denote by $B^i_\star(p,q,\kappa)$ the set of all plans in $B^i(p,q,\kappa)$ having finite equivalent payoffs.

Definition 4.3. *A competitive equilibrium with finite equivalent payoffs for the economy* $\mathcal{E}$ *is a family of prices and delivery rates* $(p,q,\kappa) \in \Xi$ *together with an allocation* $\boldsymbol{a} = (a^i)_{i\in I}$ *with* $a^i \in A$ *such that conditions (b), (c), (d) and*

(a') *for every agent i, the plan* a^i *has finite equivalent payoffs and is optimal among all budget feasible plans with finite equivalent payoffs, i.e.,*

$$a^i \in d^i_\star(p,q,\kappa) \equiv \{a \in B^i_\star(p,q,\kappa) \ : \ \text{Pref}^i(p,a) \cap B^i_\star(p,q,\kappa) = \emptyset\}$$

are satisfied.

We denote by $\text{Eq}_\star(\mathcal{E})$ the set of competitive equilibria with finite equivalent payoffs for the economy $\mathcal{E}$. We propose to compare our equilibrium concept with the one found in Araujo et al. (2002).

4.3 No default penalty

We consider the case where collateral repossession is the only enforcement mechanism and that default penalties are equal to zero as in Araujo et al. (2002) and Kubler and Schmedders (2003). One may expect $B^i_\star(p,q,\kappa)$ to be a strict subset of $B^i(p,q,\kappa)$. However, as the following proposition shows, the two sets coincide. In fact, in the model proposed by Araujo et al. (2002), any budget feasible allocation with a finite utility has finite equivalent payoffs. This is a consequence of the absence of default penalties or explicit economic punishments.

Proposition 4.2. Assume that there is no default penalty and let $a = (x,\theta,\varphi,d)$ be a plan in the budget set $B^i(p,q,\kappa)$. If $U^i(x)$ is finite then a has finite equivalent payoffs, i.e., a belongs to $B^i_\star(p,q,\kappa)$.

Proof of Proposition 4.2. Fix an agent i and consider a budget feasible plan $a \in B^i(p,q,\kappa)$ with a finite utility. Fix a period $t \geqslant 1$ and $\varepsilon > 0$. Since $U^i(x)$ is finite, there exists $T \geqslant t+1$ such that

$$\sum_{\tau\geqslant T} \sum_{\xi\in D_\tau} u^i(\xi, x(\xi)) \leqslant \varepsilon.$$

Consider now the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) < T \\ \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T \\ \\ (\omega^i(\xi), 0, 0, 0) & \text{if} \quad t(\xi) > T \end{cases}$$

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also Zhang (1997), Alvarez and Jermann (2000) and Kehoe and Levine (2001)) requires the agency or the agents to know the other agents' characteristics. For instance, under self-monitoring, lenders will only provide credit to the extend that they are able to calculate the borrowers' expected discounted lifetime utility from participating in the asset markets and their corresponding utility in autarky.

where

$$\forall \xi \in D_T, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Observe that the plan $\widehat{a}$ is budget feasible, belongs to C^T and satisfies

$$\widehat{a}\mathbf{1}_{[0,T-1]} = a\mathbf{1}_{[0,T-1]}.$$

Fix $\tau \geqslant T$. Since $T-1 \geqslant t$, in order to prove that the plan a has finite equivalent payoffs, we need to compare $U^{i,\tau}(\widehat{x})$ and $U^{i,\tau}(x)$. Observe that

$$\begin{aligned} U^{i,\tau}(\widehat{x}) &= U^{i,T-1}(x) + \sum_{T \leqslant s \leqslant \tau} \sum_{\xi \in D_s} u^i(\xi, \omega^i(\xi)) \\ &\geqslant U^{i,T-1}(x) \\ &\geqslant U^{i,\tau}(x) - \sum_{T \leqslant s \leqslant \tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \\ &\geqslant U^{i,\tau}(x) - \varepsilon. \end{aligned}$$

We have thus proved that the plan a has finite equivalent payoffs.

□

A direct implication of the last proposition is that, when there is no loss of utility in case of default, the sets $\mathrm{Eq}(\mathscr{E})$ and $\mathrm{Eq}_\star(\mathscr{E})$ coincide. This observation allows us to obtain the existence result of Araujo et al. (2002) as a direct corollary of our equilibrium existence result (see Section 5).

Proposition 4.3. If there is no default penalty then $(\pi, \boldsymbol{a})$ is a competitive equilibrium, if and only if, it is a competitive equilibrium with finite equivalent payoffs, i.e., the sets $\mathrm{Eq}(\mathscr{E})$ and $\mathrm{Eq}_\star(\mathscr{E})$ coincide.

Proof of Proposition 4.3. Let $(\pi, \boldsymbol{a}) \in \mathrm{Eq}(\mathscr{E})$ be a competitive equilibrium. Fix an agent $i \in I$. In order to prove that a^i belongs to the demand $d^i_\star(\pi)$, it is sufficient to prove that a^i has finite equivalent payoffs. Since $\boldsymbol{a}$ is feasible we have $x^i(\xi) \leqslant \Omega(\xi)$. From (H.3), we get that $U^i(x^i)$ is finite. The desired result follows from Proposition 4.2.

Now let $(\pi, \boldsymbol{a}) \in \mathrm{Eq}_\star(\mathscr{E})$ be a competitive equilibrium with finite equivalent payoffs. We only have to prove that a^i belongs to $d^i(\pi)$ for each agent i. Fix an agent i and assume by contradiction that there exists a plan a in $B^i(\pi)$ such that $U^i(x) > U^i(x^i)$. If $U^i(x)$ is finite then, applying Proposition 4.2, we get that $a \in B^i_\star(\pi)$: contradiction. Therefore, we must have $U^i(x) = \infty$, implying that there exists $T \geqslant 1$ such that

$$U^{i,T}(x) > U^i(x^i).$$

Consider the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi)=\begin{cases} a(\xi) & \text{if} \quad t(\xi)\leqslant T \\ & \\ (\omega^i(\xi),0,0,\widehat{d}(\xi)) & \text{if} \quad t(\xi)=T+1 \\ & \\ (\omega^i(\xi),0,0,0) & \text{if} \quad t(\xi)>T+1 \end{cases}$$

where

$$\forall \xi \in D_{T+1}, \quad \forall j \in J, \quad \widehat{d}(\xi,j)=D(p,\xi,j)\varphi(\xi^-,j).$$

Since the plan $\widehat{a}$ is budget feasible and belongs to C^{T+1}, it has finite equivalent payoffs and belongs to $B^i_\star(p,q,\kappa)$. Moreover we have

$$U^i(\widehat{x})=U^{i,T}(x)+\sum_{\xi\in D\setminus D^T} u^i(\xi,\omega^i(\xi))>U^i(x^i).$$

This contradicts the optimality of x^i in $B^i_\star(p,q,\kappa)$. □

5 Existence of equilibrium with finite equivalent payoffs

Levine and Zame (1996) proved that finitely effective debt constraints are compatible with equilibrium when the default penalty is infinite and no collateral is required. We argued in the previous section that a reasonable adaptation of those endogenous borrowing constraints to models with limited commitment is to restrict plans to have finite equivalent payoffs. We formally defined the concept of equilibrium with finite equivalent payoffs and we have shown its relation with respect to the equilibrium concepts found in the papers of Araujo et al. (2002) and Kubler and Schmedders (2003). In this section, we are concerned with the issue of existence of such equilibria. We show that if agents are myopic with respect to default penalties, restricting actions to have finite equivalent payoffs allows to rule out Ponzi schemes and guarantees the existence of an equilibrium. Myopia in our setting refers to the time preference of default: the disutility of defaulting today is greater than the disutility of defaulting in the distant future and vanishes in the long run. That is, myopia implies a reasonable restriction on the asymptotic behavior of default penalties. Therefore, it is more appealing than conditions imposed on the level of default penalties (see Martins-da-Rocha and Vailakis (2010)). Myopic agents have incentives to choose optimally plans that are associated with effective payments in the short run and default in the long run. That is, agents are willing to deliver more that the value of the depreciated collateral in initial periods and therefore they have incentives to require additional borrowing to serve their debt. In the long run, however, agents have no incentives to honor their debts since default does not hurt much.

Before introducing the formal definition of myopic agents with respect to default penalties, we need to introduce some notations. For each asset j and node ξ, we denote by $M(\xi,j)$ the real number

$$\min_{\ell\in L}\frac{\Omega(\xi,\ell)}{C(\xi,j,\ell)}.$$

Observe that under Assumption 3.2, we have $M(\xi,j)<\infty$. Finally, for every node $\xi\neq\xi_0$ we let[14]

$$H(\xi,j)=M(\xi^-,j)\sup_{p\in\Delta(L)}\frac{[pA(\xi,j)-pY(\xi)C(\xi^-,j)]^+}{pv(\xi)}.$$

The quantity $H(\xi,j)$ is the maximum amount in real terms that an agent may default on asset j if his plan is physically feasible. It is straightforward to verify that if a in A is a physically feasible plan and (p,q,κ) in Π is a process of prices and delivery rates, then for each node ξ and each asset j, we have

$$\varphi(\xi,j)\leqslant M(\xi,j)\quad\text{and}\quad\left[V(p,\xi,j)\varphi(\xi^-,j)-d(\xi,j)\right]^+\leqslant H(\xi,j).$$

Definition 5.1. *Agent i is said to be myopic with respect to default penalties if the disutility suffered at the initial period from defaulting in the long run is negligible, i.e.,*

$$\liminf_{T\to\infty}\sum_{\xi\in D_T}\sum_{j\in J}\lambda^i(\xi,j)H(\xi,j)=0.$$

Assuming that agents are myopic with respect to default penalties is a very mild assumption since it is automatically satisfied for every standard economy as defined below (see e.g. Araujo and Sandroni (1999)).

Definition 5.2. *The economy $\mathcal{E}$ is said standard if Assumptions 3.1 and 3.2 are satisfied and if for each agent i there exist*

- **(S.1)** *a discount factor $\beta_i\in(0,1)$;*
- **(S.2)** *a sequence $(P^i_t)_{t\geqslant 1}$ of beliefs about nodes at period t represented by a probability $P^i_t\in\operatorname{Prob}(D_t)$;*
- **(S.3)** *an instantaneous felicity function $v^i: D\times\mathbb{R}^L_+\to[0,\infty)$;*
- **(S.4)** *an instantaneous default penalty $\mu^i(\xi,j)\in(0,\infty)$ for each node $\xi>\xi_0$;*

such that for each node $\xi\in D$,

$$u^i(\xi,\cdot)=[\beta_i]^{t(\xi)}P^i_{t(\xi)}(\xi)v^i(\xi,\cdot)$$

for each $j\in J$,

$$\lambda^i(\xi,j)=[\beta_i]^{t(\xi)}P^i_{t(\xi)}(\xi)\mu^i(\xi,j)$$

and the processes $(A(\xi,j))_{\xi>\xi_0}$, $(\mu^i(\xi,j))_{\xi>\xi_0}$ and $(G(\xi,j))_{\xi\in D}$ are uniformly bounded from above, where

$$G(\xi,j)=\frac{1}{\max_{\ell\in L}C(\xi,j,\ell)}.$$

---

[14] The set $\Delta(L)$ is the simplex in $\mathbb{R}^L_+$, i.e., $\Delta(L)=\{p\in\mathbb{R}^L_+ : \sum_{\ell\in L}p(\ell)=1\}$.

When agents are myopic with respect to default penalties, any budget and physically feasible plan $a \in B^i(p,q,\kappa) \cap \mathrm{F}^i$ has actually finite equivalent payoffs. This result will turn out to be crucial in the process of proving the existence of an equilibrium with finite equivalent payoffs.

Proposition 5.1. If agent i is myopic with respect to default penalties, then every budget and physically feasible plan has finite equivalent payoffs. In other words, we have

$$B^i(p,q,\kappa) \bigcap \mathrm{F}^i \subset B^i_\star(p,q,\kappa).$$

Proof of Proposition 5.1. Fix an agent i and consider a plan a that is budget and physically feasible, i.e., $a \in B^i(p,q,\kappa) \cap \mathrm{F}^i$. Fix a period $t \geqslant 1$ and $\varepsilon > 0$. Since the allocation a is physically feasible, we have $x(\xi) \leqslant \Omega(\xi)$, implying that

$$\sum_{\xi \in D} u^i(\xi, x(\xi)) < \infty.$$

Therefore there exists $T^0 \geqslant 1$ such that

$$\sum_{T \geqslant T^0} \sum_{\xi \in D_T} u^i(\xi, x(\xi)) \leqslant \frac{\varepsilon}{2}.$$

Since agent i is myopic with respect to default penalties, there exists $T > \max\{t, T^0\}$ such that

$$\sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi, j) H(\xi, j) \leqslant \frac{\varepsilon}{2}.$$

Consider now the plan $\widehat{a}$ defined by

$$\widehat{a}(\xi) = \begin{cases} a(\xi) & \text{if} \quad t(\xi) < T \\ (\omega^i(\xi), 0, 0, \widehat{d}(\xi)) & \text{if} \quad t(\xi) = T \\ (\omega^i(\xi), 0, 0, 0) & \text{if} \quad t(\xi) > T \end{cases}$$

where

$$\forall \xi \in D_T, \quad \forall j \in J, \quad \widehat{d}(\xi, j) = D(p, \xi, j)\varphi(\xi^-, j).$$

Observe that the plan $\widehat{a}$ satisfies

$$\widehat{a} \in B^i(p,q,\kappa) \cap C^T \quad \text{and} \quad \widehat{a}\mathbf{1}_{[0,T-1]} = a\mathbf{1}_{[0,T-1]}.$$

Moreover, for every $\tau \geqslant T$ we have

$$\begin{aligned}
\Pi^{i,T}(p,\widehat{a}) &= \Pi^{i,T-1}(p,\widehat{a}) \\
&+ \sum_{\xi \in D_T} \left[u^i(\xi, \omega^i(\xi)) - \sum_{j \in J} \lambda^i(\xi,j) \frac{[V(p,\xi,j) - D(p,\xi,j)]\ \varphi(\xi^-,j)}{p(\xi)v(\xi)} \right] \\
&\geqslant \Pi^{i,T-1}(p,a) - \sum_{\xi \in D_T} \sum_{j \in J} \lambda^i(\xi,j) H(\xi,j) \\
&\geqslant \Pi^{i,T-1}(p,a) - \frac{\varepsilon}{2} \\
&\geqslant \Pi^{i,\tau}(p,a) - \frac{\varepsilon}{2} - \sum_{T \leqslant s \leqslant \tau} \sum_{\xi \in D_s} u^i(\xi, x(\xi)) \\
&\geqslant \Pi^{i,\tau}(p,a) - \varepsilon.
\end{aligned}$$

It follows that for every $\tau \geqslant T$

$$\Pi^{i,\tau}(p,\widehat{a}) = \Pi^{i,T}(p,\widehat{a}) + \sum_{\xi \in D^\tau \setminus D^T} u^i(\xi, \omega^i(\xi)) > \Pi^{i,\tau}(p,a) - \varepsilon.$$

Since $T - 1 \geqslant t$, this implies that the plan a has finite equivalent payoffs. ☐

Remark 5.1. Given Proposition 5.1 one may wonder whether restricting plans to have finite equivalent payoffs is relevant to the issue of existence. Since myopia implies that budget and physically feasible plans have finite equivalent payoffs, why one should impose any kind of debt constraints on available plans to ensure existence? The answer to this question lies on the fact that in decentralized economies agents do not take into account feasibility restrictions when they solve their maximization problem. Only budgetary restrictions are relevant for them. But if this is the case, in the absence of borrowing constraints, agents can run a Ponzi scheme and equilibria may fail to exist.[15]

The main contribution of this paper is the following existence result. The proof is postponed to the appendix.

Theorem 5.1. *If every agent is myopic with respect to default penalties then a competitive equilibrium with finite equivalent payoffs exists, i.e.,* $\mathrm{Eq}_\star(\mathscr{E}) \neq \emptyset$.

Given Proposition 4.3, we can obtain the main existence result in Araujo et al. (2002, Theorem 2) as a corollary of Theorem 5.1.

Corollary 5.1 (Araujo et al. (2002)). *If there is no default penalty then there exists a competitive equilibrium, i.e.,* $\mathrm{Eq}(\mathscr{E}) \neq \emptyset$.

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[15] Martins-da-Rocha and Vailakis (2010) provide an example of an economy with myopic agents and no borrowing constraints in which equilibria fail to exist.

Remark 5.2. Our aim is to illustrate the role of our proposed borrowing constraints in a familiar framework with limited commitment and possible default at equilibrium. In that respect, we have chosen to model agents' payoffs as in Dubey et al. (2005) and Páscoa and Seghir (2009). That is, each agent's payoff equals the utility for consumption minus the default penalty assumed to be a linear function of the amount of default expressed in real terms. We could alternatively consider a more general framework where the payoff $\Pi^{i,t}(p,a)$ is described as follows

$$\Pi^{i,t}(p,a) \equiv \sum_{\xi \in D^t} \Pi^i(\xi, x(\xi), \delta(p(\xi), \varphi(\xi^-), d(\xi)))$$

where

$$\delta(p(\xi), \varphi(\xi^-), d(\xi)) = (\delta_j(p(\xi), \varphi(\xi^-), d(\xi)))_{j \in J}$$

with

$$\delta_j(p(\xi), \varphi(\xi^-), d(\xi)) \equiv \frac{[V(p,\xi,j)\varphi(\xi^-,j) - d(\xi,j)]^+}{p(\xi)v(\xi)}$$

and $\Pi^i(\xi,\cdot,\cdot)$ is a function from $\mathbb{R}^L_+ \times \mathbb{R}^J_+$ to $[-\infty,\infty)$. The number $\Pi^i(\xi,x,\delta)$ represents the payoff received (or felt) by agent i at $t=0$ if at node ξ, he consumes the bundle $x \in \mathbb{R}^L_+$ and defaults on each asset j the quantity $\delta_j \geqslant 0$ expressed in real terms. Observe that the model considered in this paper is a particular case where

$$\Pi^i(\xi,x,\delta) = u^i(\xi,x) - \sum_{j \in J} \lambda^i(\xi,j)\delta_j.$$

We can replace Assumptions H.2 and H.3 by the conditions

(G.2) for every node ξ, the function $(x,\delta) \mapsto \Pi^i(\xi,x,\delta)$ is concave, continuous, strictly increasing on x, decreasing on δ with $\Pi^i(\xi,0,0)=0$;

(G.3) the infinite sum $\sum_{\xi \in D} \Pi^i(\xi,\Omega(\xi),0)$ is finite.

If we replace the condition that each agent i is myopic with respect to default penalties by the following one

$$\liminf_{T\to\infty} \sum_{\xi \in D_T} \Pi^i(\xi,0,H(\xi)) = 0$$

then we can reproduce all the arguments of the paper.[16]

6 Conclusion

What makes general equilibrium models with collateral such as those in Araujo et al. (2002) and Kubler and Schmedders (2003) very appealing is that collateral constraints not only do exist in actual markets but seem to be an efficient mechanism to preclude

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[16] In particular, Propositions 5.1 and A.1 hold true with generalized payoffs.

Ponzi schemes without imposing any *ad-hoc* constraint on debt. The recent contributions of Páscoa and Seghir (2009) and Revil and Torres-Martínez (2010) show that the positive results in Araujo et al. (2002) may not be robust: the effectiveness of collateral requirements to bound debt may not be valid anymore in the natural case where there are other mechanisms leading agents to overpay, that is, to repay more than the collateral when the value of their debt actually exceeds the collateral value.

We argue that endogenous debt constraints à la Levine and Zame (1996) are still relevant to restore equilibrium in models with collateralized promises and default penalties. Assuming that each agent conceives that at any time period he may be restricted to have access to credit markets for a finite number of periods, we introduce in the setting of Araujo et al. (2002), Kubler and Schmedders (2003) and Páscoa and Seghir (2009) the concept of plans with *finite equivalent payoffs*. When payments are fully enforced, our concept of plans with finite equivalent payoffs coincides with the concept of plans with finitely effective debt introduced by Levine and Zame (1996). When there are collateral requirements but no default penalties, any budget feasible plan has automatically finite equivalent payoffs. In particular, our budget set coincides with the standard one defined in Araujo et al. (2002) and Kubler and Schmedders (2003). Assuming a mild assumption on default penalties, namely that agents are myopic with respect to default penalties, we show that restricting actions to have finite equivalent payoffs rules out Ponzi schemes and guarantees equilibrium existence while keeping the minimal ability to borrow and lend that we expect in our model. The proof is very simple and intuitive. In particular, the main existence result in Araujo et al. (2002) is a direct corollary of our existence result.

Appendix

A Proof of Theorem 5.1

Fix $\tau \in \mathscr{T}$ with $\tau > 0$. Recall that A^τ denotes the set of all plans $a \in A$ such that

$$\forall \xi \in D, \quad t(\xi) > \tau \Longrightarrow a(\xi) = 0.$$

We let B^τ be the set of plans $a \in A^\tau$ satisfying the additional condition

$$\forall \xi \in D, \quad t(\xi) = \tau \Longrightarrow \varphi(\xi) = 0.$$

Given a process $(p, q, \kappa) \in \Xi$, we denote by $B^{i,\tau}(p, q, \kappa)$ the set defined by

$$B^{i,\tau}(p, q, \kappa) \equiv B^i(p, q, \kappa) \cap B^\tau.$$

Definition A.1. *A competitive equilibrium for the truncated economy $\mathscr{E}^\tau$ is a family of prices and delivery rates $\pi = (p, q, \kappa) \in \Xi$ and an allocation $\boldsymbol{a} = (a^i)_{i \in I}$ with $a^i \in B^\tau$ such that*

(a) *for every agent i, the plan a^i is optimal, i.e.,*

$$a^i \in d^{i,\tau}(p,q,\kappa) \equiv \operatorname{argmax}\{\Pi^{i,\tau}(p,a) \ : \ a \in B^{i,\tau}(p,q,\kappa)\}; \tag{A.1}$$

(b) *commodity markets clear at every node up to period τ, i.e.,*

$$\sum_{i\in I} x^i(\xi_0) = \sum_{i\in I} \omega^i(\xi_0) \tag{A.2}$$

and for all $\xi \in D^\tau \setminus \{\xi_0\}$,

$$\sum_{i\in I} x^i(\xi) = \sum_{i\in I} \left[\omega^i(\xi) + Y(\xi)x^i(\xi^-)\right]; \tag{A.3}$$

(c) *asset markets clear at every node up to period $\tau - 1$, i.e., for all $\xi \in D^{\tau-1}$,*

$$\sum_{i\in I} \theta^i(\xi) = \sum_{i\in I} \varphi^i(\xi); \tag{A.4}$$

(d) *deliveries match up to period τ, i.e., for all $\xi \in D^\tau \setminus \{\xi_0\}$ and all $j \in J$,*

$$\sum_{i\in I} V(\kappa,p,\xi,j)\theta^i(\xi^-,j) = \sum_{i\in I} d^i(\xi,j). \tag{A.5}$$

Remark A.1. Observe that if a plan a belongs to B^τ, then $\Pi^{i,\tau}(p,a)$ and $\Pi^i(p,a)$ coincide for every price process p.

Remark A.2. Observe that if $(\pi, \boldsymbol{a})$ is a competitive equilibrium for the truncated economy $\mathscr{E}^\tau$, then without any loss of generality, we can assume that $q(\xi) = 0$ and $\theta(\xi) = 0$ for every terminal node $\xi \in D_\tau$.

It is claimed in Páscoa and Seghir (2009) that a competitive equilibrium for every truncated economy $\mathscr{E}^\tau$ exists, and that commodity prices are uniformly bounded away from 0. The interested reader can find a detailed proof of the following result in the appendix of Martins-da-Rocha and Vailakis (2010).

Proposition A.1. There exists a process $m = (m(\xi))_{\xi\in D}$ of strictly positive numbers $m(\xi) > 0$ such that for every period τ, there exists a competitive equilibrium $(\pi^\tau, \boldsymbol{a}^\tau)$ of the truncated economy $\mathscr{E}^\tau$ satisfying $\left\|p^\tau(\xi)\right\| \geqslant m(\xi)$ at every node $\xi \in D^{\tau-1}$.

For each $\tau \in \mathscr{T}$ with $\tau \geqslant 1$, we let $(\pi^\tau, \boldsymbol{a}^\tau)$ be a competitive equilibrium for the truncated economy $\mathscr{E}^\tau$ where $\pi^\tau = (p^\tau, q^\tau, \kappa^\tau)$ and $\boldsymbol{a}^\tau = (a^{i,\tau})_{i\in I}$. Each process π^τ belongs to $\operatorname{cl}\Xi$ which is weakly compact as a product of compact sets. Passing to a subsequence if necessary, we can assume that the sequence $(\pi^\tau)_{\tau\in\mathscr{T}}$ converges to a process $\pi = (p,q,\kappa)$ in $\operatorname{cl}\Xi$. Following Proposition A.1, for each node $\xi \in D$, we have $\left\|p(\xi)\right\| \geqslant m(\xi) > 0$. In particular, for each period t and every plan $a \in A$, the payoff $\Pi^{i,t}(p,a)$ is well defined.

By feasibility at each node ξ, we get for each j

$$x^{i,\tau}(\xi) \leqslant \Omega(\xi), \quad \varphi^{i,\tau}(\xi,j) \leqslant M(\xi,j) \quad \text{and} \quad \theta^{i,\tau}(\xi,j) \leqslant M(\xi,j).$$

This implies that the sequence $(x^{i,\tau}(\xi), \varphi^{i,\tau}(\xi), \theta^{i,\tau}(\xi))_{\tau\in\mathscr{T}}$ is uniformly bounded. By optimality, the delivery $d^{i,\tau}(\xi,j)$ is always lower than $V(p^\tau,\xi,j)\varphi^{i,\tau}(\xi^-,j)$ and therefore the sequence $(d^{i,\tau}(\xi))_{\tau\in\mathscr{T}}$ is uniformly bounded. Passing to a subsequence if necessary, we can assume that for each i, the sequence $(a^{i,\tau})_{\tau\in\mathscr{T}}$ converges to a process $a^i \in A$.

We claim that $(\pi, \boldsymbol{a})$ is a competitive equilibrium with finite equivalent payoffs for the economy $\mathscr{E}$. It is straightforward to check that each plan a^i belongs to the budget set $B^i(p,q,\kappa)$ and that the feasibility conditions (2.5), (2.6), (2.7) and (2.8) are satisfied. Applying Proposition 5.1, we get that the plan a^i has finite equivalent payoffs. We propose now to prove that a^i is optimal among plans with finite equivalent payoffs, i.e., $\mathrm{Pref}^i(p,a^i) \cap B^i_\star(p,q,\kappa)$ is empty. Assume by way of contradiction that there exists a plan $\overline{a}$ in the budget set $B^i_\star(p,q,\kappa)$, $\varepsilon > 0$ and $T^1 \in \mathbb{N}$ satisfying

$$\forall T \geqslant T^1, \quad \Pi^{i,T}(p,\overline{a}) > \Pi^{i,T}(p,a^i) + \varepsilon. \tag{A.6}$$

Since a^i is physically feasible, we have

$$\forall \xi \in D, \quad x^i(\xi) \leqslant \Omega(\xi).$$

It follows from Assumptions (H.2) and (H.3) that

$$U^i(x^i) \leqslant U^i(\Omega) < +\infty.$$

This implies that

$$\Pi^i(p,a^i) \equiv \lim_{T\to\infty} \Pi^{i,T}(p,a^i)$$

exists in $[-\infty,\infty)$. It follows that there exists $T^2 \geqslant T^1$ such that

$$\forall T \geqslant T^2, \quad \Pi^{i,T}(p,a^i) + \frac{\varepsilon}{2} > \Pi^i(p,a^i). \tag{A.7}$$

Since the plan $\overline{a}$ has finite equivalent payoffs, there exists $T > T^2$ and $\widetilde{a}$ in the set $B^i(p,q,\kappa) \cap C^T$ such that

$$\widetilde{a}\mathbf{1}_{[0,T^2]} = \overline{a}\mathbf{1}_{[0,T^2]} \quad \text{and} \quad \inf_{\tau\geqslant T} \left[\Pi^{i,\tau}(p,\widetilde{a}) - \Pi^{i,\tau}(p,\overline{a})\right] \geqslant -\frac{\varepsilon}{4}. \tag{A.8}$$

We denote by $\widehat{a}$ the plan defined by

$$\forall \xi \in D, \quad \widehat{a}(\xi) = \begin{cases} \widetilde{a}(\xi) & \text{if} \quad t(\xi) \leqslant T \\ (0,0,0,0) & \text{if} \quad t(\xi) > T. \end{cases}$$

Observe that $\widehat{a}$ belongs to the truncated budget set $B^i(p,q,\kappa) \cap B^T$ and satisfies

$$\widehat{a}\mathbf{1}_{[0,T^2]} = \overline{a}\mathbf{1}_{[0,T^2]} \quad \text{and} \quad \Pi^{i,T}(p,\widehat{a}) \geqslant \Pi^{i,T}(p,\overline{a}) - \frac{\varepsilon}{4}. \tag{A.9}$$

Combining (A.6), (A.7) and (A.9) we get

$$\Pi^{i,T}(p,\widehat{a}) > \Pi^i(p,a^i) + \frac{\varepsilon}{4}.$$

We let ψ^i be the correspondence from A to A^T defined by

$$\forall a \in A, \quad \psi^i(a) = \left\{ b \in B^T \ : \ \Pi^{i,T}(p,b) > \frac{\varepsilon}{4} + \Pi^i(p,a) \right\}.$$

Let F^i be the correspondence from $\Xi \times A$ to A^T defined by

$$\forall (\pi,a) \in \Xi \times A, \quad F^i(\pi,a) = B^{i,T}(\pi) \cap \psi^i(a).$$

Following the arguments in Páscoa and Seghir (2009), we have the following continuity result.

Lemma A.1. *The correspondence F^i is lower semi-continuous on $\Xi \times A$ for product topologies.*

Observe that

$$\widehat{a} \in F^i((p,q,\kappa),a^i).$$

We proved that there exists a strictly increasing sequence $(T_n)_{n\in\mathbb{N}}$ with $T_n \in \mathbb{N}$ such that

$$\lim_{n\to\infty}((p_n,q_n,\kappa_n),a^i_n) = ((p,q,\kappa),a^i)$$

where

$$(p_n,q_n,\kappa_n) = (p^{T_n},q^{T_n},\kappa^{T_n}) \quad \text{and} \quad a^i_n = a^{i,T_n}.$$

Since F^i is lower semi-continuous, there exists $\nu \in \mathbb{N}$ large enough such that $T_\nu \geqslant T$ and the set $F^i((p_\nu,q_\nu,\kappa_\nu),a^i_\nu)$ is non-empty. Let $\widehat{a}_\nu$ be an element of that set. This means that

$$\widehat{a}_\nu \in B^{i,T}(p_\nu,q_\nu,\kappa_\nu) \quad \text{and} \quad \Pi^{i,T}(p_\nu,\widehat{a}_\nu) \geqslant \Pi^i(p_\nu,a^i_\nu) + \frac{\varepsilon}{4}.$$

Since $T_\nu \geqslant T$, we have

$$B^{i,T}(p_\nu,q_\nu,\kappa_\nu) \subset B^{i,T_\nu}(p_\nu,q_\nu,\kappa_\nu) \quad \text{and} \quad \Pi^{i,T_\nu}(p_\nu,\widehat{a}_\nu) \geqslant \Pi^{i,T}(p_\nu,\widehat{a}_\nu).$$

It follows that

$$\Pi^{i,T_\nu}(p_\nu,\widehat{a}_\nu) > \Pi^i(p_\nu,a^i_\nu) = \Pi^{i,T_\nu}(p_\nu,a^i_\nu).$$

This contradicts the optimality of a^i_ν.[17]

We have thus proved that for each i, the plan a^i has finite equivalent payoffs and satisfies

$$\text{Pref}^i(p,a^i) \cap B^i_\star(p,q,\kappa) = \emptyset.$$

This means that a^i belongs to the demand set $d^i_\star(\pi)$. We already proved that all markets clear. This means that $(\pi,\boldsymbol{a})$ is a competitive equilibrium with finite equivalent payoffs.

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[17] Recall that $((p_\nu,q_\nu,\kappa_\nu),\boldsymbol{a}_\nu)$ is a competitive equilibrium of the truncated economy $\mathscr{E}^{T_\nu}$.

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