Πανεπιστήμιο Κύπρου

ΤΜΗΜΑ ΦΥΣΙΚΗΣ

ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

Εντροπία σύμπλεξης σε σκεδάσεις QED και βαρύτητας στο παράδοξο διατήρησης της πληροφορίας στις μαύρες τρύπες

Αναστάσιος Ηρακλέους

Επιβλέπων καθηγητής: Νικόλαος Τούμπας

ΜΑΪΟΣ 2020

ΠΑΝΕΠΙΣΤΗΜΙΟ ΚΥΠΡΟΥ

ΤΜΗΜΑ ΦΥΣΙΚΗΣ

ΔΙΠΛΩΜΑΤΙΚΗ ΕΡΓΑΣΙΑ

Εντροπία σύμπλεξης σε σκεδάσεις QED και βαρύτητας στο παράδοξο διατήρησης της πληροφορίας στις μαύρες τρύπες

Αναστάσιος Ηρακλέους

Επιβλέπων καθηγητής: Νικόλαος Τούμπας

H diplwmatik ergasÐa upobl jhke proc merik ekpl rwsh twn apait sewn apìkthshc tou ptuqÐou Fusik c sto Tm ma Fusik c tou PanepisthmÐou KÔprou

ΜΑΪΟΣ 2020

Perieqìmena

Perieqìmena i

EuqaristÐec vi

Prìlogoc vii

I MaÔrec trÔpec, par˘doxa kai IR fainìmena 1

1 MaÔrec trÔpec 2
1.1 LÔsh Schwarzschild 2
1.1.1 Suntetagmènec tortoise 3
1.1.2 Suntetagmènec Rindler 4
1.1.3 Suntetagmènec Kruskal-Szekeres 4
1.2 Diagr˘mmata Penrose 5
1.2.1 Qwrìqronoc Minkowski 6
1.2.2 Qwrìqronoc Schwarzschild 7
1.3 Metrik Vaidya 8
1.4 Fortismènec maÔrec trÔpec 10

2 Jermodunamik pedÐwn se kampulwmènouc qwrìqronouc 14
2.1 PedÐa se kampulwmènouc qwrìqronouc 14
2.1.1 Klassik˘ bajmwt˘ pedÐa sthn gewmetrÐa Schwarzschild 14
2.1.2 Klasik˘ bajmwt˘ pedÐa sthn gewmetrÐa Ridler 16
2.2 Sun˘rthsh epimerismoÔ kbantik¸n pedÐwn 18
2.2.1 Sun˘rthsh epimerismoÔ me sunarthsiak˘ oloklhr¸mata 19
2.2.2 Jemeli¸dhc kat˘stash sust matoc 20
2.3 JermokrasÐa pedÐwn se kampulwmènouc qwrìqronouc 21
2.3.1 Qwrìqronoc Rindler 21
2.3.2 Qwrìqronoc Schwarzschild 22
2.4 EntropÐa sÔmplexhc 24
2.5 Jermik entropÐa 25

2.5.1 Par^deigma 26
2.6 EntropÐa sÔmplexhc sto qwrìqrono Rindler kai sth maÔrh trÔpa 26
2.7 Enèrgeia Rindler maÔrhc trÔpac 28
2.8 Qrìnoc zw c maÔrhc trÔpac 28
2.9 Hlektrikèc idiìthtec orÐzonta maÔrhc trÔpac 29

3 AktinobolÐa, upèrujra jewr mata kai diathroÔmena reÔmata sthn QED 32
3.1 Diorj¸seic aktinobolÐac sthn QED 32
3.1.1 UpologismoÐ se brìqouc 34
3.2 Upèrujra jewr mata sthn QED kai sth barÔthta 38
3.2.1 Eisagwg enìc upèrujrou fwtonÐou barutonÐou 38
3.2.2 N upèrujra swmatÐdia 39
3.2.3 ApoklÐseic lìgw eikonik¸n upèrujrwn swmatidÐwn 39
3.2.4 ApoklÐseic lìgw ekpomp c pragmatik¸n upèrujrwn swmatidÐwn 42
3.2.5 Upèrujra swmatÐdia sunolik c enèrgeiac mikrìterhc apì E 43
3.2.6 AnaÐresh apoklÐsewn 44
3.3 Meg^loi metasqhmatismoÐ bajmÐdac (LGT) 44

4 Par^doxa pou sunant^me stic maÔrec trÔpec 47
4.1 FusikoÐ Nìmoi 47
4.1.1 Diat rhsh thc kbantik c plhroforÐac 48
4.1.2 Arq thc IsodunamÐac 48
4.1.3 Mh klwnopoÐhsh kbantik c plhroforÐac 48
4.1.4 Arq thc sumplhrwmatikìthtac 49
4.1.5 Arq thc olografÐac 50
4.2 Par^doxo klwnopoÐhshc plhroforÐac 50
4.3 Diat rhsh baruonikoÔ arijmoÔ 52
4.4 Par^doxo diat rhshc plhroforÐac kat^ thn exaôlwsh miac maÔrhc trÔpac . 54
4.4.1 Proèleush paradìxou 54
4.4.2 Prot^seic gia lÔsh tou paradìxou 55
4.4.2.1 Prìtash Page 55
4.4.2.2 ParabÐash kbantomhqanik c 55
4.4.2.3 SÔmplexh metaxÔ uperÔjrwn kai uperiwd¸n bajm¸n eleujerÐac 55

II Entropĺa sÔmplexhc se sked^seic fortismènwn swmatidĺwn 58

Eisagwg 59

5 Ntumènec katast^seic 62
5.1 Sumb^seic kai sumbolismìc 62
5.2 Katast^seic Faddeev-Kulish 63
5.3 Ntumèna hlektrìnia 63
5.3.1 Fusikèc idiìthtec nèfouc 64
5.4 Genikèc katast^seic 65
5.5 Pĺnakac skèdashc Faddeev-Kulish 67
5.5.1 Paragwg enìc upèrujrou fwtonĺou 69
5.6 Diakritopoĺhsh 71
5.7 'Iqnoc wc proc upèrujrouc bajmoÔc eleujerĺac 72

6 Skèdash me ntumènec katast^seic kai entropĺa sÔmplexhc 74
6.1 Pĺnakac puknìthtac twn uperiwd¸n bajm¸n eleujerĺac 76
6.2 Upologismìc thc t^xhc thc entropĺac sÔmplexhc 77
6.3 Diataraktik an^lush 80
6.4 KÔria t^xh entropĺac sÔmplexhc 85
6.5 Suneqèc ìrio 86
6.5.1 Pl^toc iM kÔriac t^xhc 88

7 Entropĺa sÔmplexhc deÔterhc t^xhc 91
7.1 'Iqnh gumn¸n katast^sewn wc proc touc upèrujrouc bajmoÔc eleujerĺac . 91
7.2 Pĺnakac puknìthtac 93
7.3 Entropĺa sÔmplexhc 96
7.3.1 'Iqnh wc proc upèrujrouc bajmoÔc eleujerĺac 96
7.3.2 An^ptugma dun^mewn tou pĺnaka puknìthtac 99
7.3.3 Entropĺec Renyi 100
7.4 To apoklĺnon mèroc entropĺac sÔmplexhc 102

Epĺlogoc 104

Parart mata 106

A Tanustèc 107

B Genik sqetikìthta 109
B.1 Exis¸seic Eistein 109
B.2 Exis¸seic kĺnhshc swmatidĺwn 111

C Kbantik jewrÐa pedÐou 112
C.1 Bajmwt^ pedÐa 112
C.2 Kanonik kb^ntwsh 113
C.3 Kb^ntwsh mèsw sunarthsiak¸n oloklhrwm^twn 114
C.3.1 EukleÐdeia sunarthsiak^ oloklhr¸mata 115
C.3.2 Par^deigma armonikoÔ talantwt 115
C.4 Je¸rhma thc Noether 119
C.4.1 Tanust c enèrgeiac-orm c 120
C.5 PedÐo Klein-Gordon 120
C.5.1 Klasikì pedÐo Klein-Gordon 120
C.5.2 Kb^ntwsh pedÐou Klein-Gordon 121
C.5.3 Diadìthc Klein-Gordon 123
C.6 PedÐo Dirac 124
C.6.1 Klasik exÐswsh Dirac 124
C.6.2 Kb^ntwsh pedÐou Dirac 125
C.6.3 Diadìthc Feynman gia to pedÐo Dirac 127
C.7 Kanìnec Feynman gia thn kbantik hlektrodunamik 127
C.7.1 QED 128
C.7.2 PÐnakac skèdashc S 129
C.7.3 Upologismìc tou pÐnaka skèdashc apì diagr^mmata Feynman . . . 130

D PÐnakec puknìthtac 132

E Om^dec sthn fusik 134
E.1 OrismoÐ 134
E.2 Om^da peristrof¸n 134
E.3 Om^da Lorentz 135

F Paramètroi Feynman 138

G Suntelestèc ìrwn stic entropÐec Renyi 140

BibliografÐa 142

Euqaris tÐec

Ja jela na euqarist sw idiaÐtera ton Dr. Nikìlao ToÔmpa gia thn kajod ghsh tou katˆ thn ekpìnhsh thc ergasÐac aut c. EpÐshc ja jela na euqarist sw to tm ma Fusik c gia thn st rixh tou.

Prìlogoc

H ^poyh pou epikrateÐ s mera metaxÔ twn jewrhtik¸n fusik¸n eÐnai ìti h fÔsh upakoÔei se sugkekrimènouc nìmouc, kai me b^sh autoÔc touc nìmouc prèpei na broÔme mia jewrÐa pou na perigr^fei ta parathroÔmena fainìmena. Me b^sh touc nìmouc thc eidik c sqetikìthta kai thc kbantomhqanik c èqei dhmiourghjeÐ to montèlo tou kajierwmènou protÔpou, pou basÐzetai sthn kbantik jewrÐa pedÐou, kai an kai den èqei akìma oloklhrwjeÐ deÐqnei na perigr^fei arket^ kal^ thn sumperifor^ thn fÔshc. 'Omwc autì to montèlo perilamb^nei mìno tic treic jemeli¸dhc allhlepidr^seic (thn isqur , thn asjen kai thn hlektromagnhtik allhlepÐdrash), all^ ìqi thn barÔthta, h opoÐa ìpwc faÐnetai den mporeÐ na perigrafeÐ apì mia kbantik jewrÐa pedÐou. O lìgoc eÐnai ìti an prospaj soume na kbant¸soume thn barÔthta sunant^me apeirismoÔc. 'Otan h èntash thc barÔthta gÐnei polÔ meg^lh, h klassik perigraf pou dÐnetai apì thn genik sqetikìthta paÔei na perigr^fei kal^ to sÔsthma. Stic ^llec allhlepidr^seic h jewrÐa mporeÐ na kanonikopoihjeÐ mèsw thc diadikasÐac thc epanakanonikopoÐhshc (renormalization) kai na d¸sei peperasmèna parathroÔmena megèjh. Sthn perÐptwsh thc barÔthtac ìmwc oi apeirismoÐ den mporoÔn na apaleifjoÔn.

An kai up^rqoun peperasmènec jewrÐec sumbatèc me touc nìmouc thc kbantomhqanik c pou perigr^foun kai tic tèsseric allhlepidr^seic (ìpwc h jewrÐa twn qord¸n), ènac trìpoc na melet soume sust mata me isqur barÔthta eÐnai na jèsoume èna ìrio sto pìso meg^lec timèc mporeÐ na p^rei h enèrgeia kai na exet^soume pwc sumperifèretai to sÔsthma. 'Etsi epitrèpoume sto sÔsthma na èqei arket^ isqur barÔthta ¸ste oi diorj¸seic thc klassik c jewrÐac (kbantik^ fainìmena) na eÐnai aisjhtèc, ìmwc ìqi aujaÐreta isqur gia na mporoÔme na perigr^youme to sÔsthma qrhsimopoi¸ntac thn kbantik jewrÐa pedÐou. Aut h prosèggish onom^zetai hmiklasik . Se aut thn prosèggish apaitoÔme na isqÔoun oi nìmoi thc genik c sqetikìthtac. Se autèc tic klÐmakec enèrgeiac ìpwc kai se ìlec tic klÐmakec enèrgeiac xeqwrist^ oi nìmoi thc fÔshc stouc opoÐouc basÐzetai h jewrÐa prèpei na eÐnai autosunepeÐc, dhlad na mhn parousi^zontai par^doxa. Diaforetik^ h jewrÐa den eÐnai apodekt . Gi' autì eÐnai shmantik h epÐlush twn paradìxwn se ìlec tic klÐmakec enèrgeiac.

Sthn hmiklasik prosèggish èna apì ta pio shmantik^ par^doxa pou sunant^me eÐnai to par^doxo thc diat rhshc thc plhroforÐac kat^ thn exaölwsh miac maÔrhc trÔpac, pou fèrnei se sÔgkroush tic arqèc thc kbantomhqanik c me tic arqèc thc genik c sqetikìthtac.

To par^doxo autì èqei diatupwjeÐ apì ton jewrhtikì fusikì Stephen Hawking. SÔmfwna me to par^doxo autì, kat^ thn exaölwsh miac maÔrhc trÔpac h plhroforÐa pou up rqe arqik^ sto sÔsthma kai èpese mèsa sthn maÔrh trÔpa q^netai ìtan h maÔrh trÔpa exaôlw-jeÐ, afoÔ sÔmfwna me thn genik sqetikìthta h plhroforÐa den mporeÐ na exèljei apì thn maÔrh trÔpa. Autì ìmwc parabi^zei touc nìmouc thc kbantomhqanik c sÔmfwna me touc opoÐouc h plhroforÐa prèpei na diathreÐtai. Den èqei up^rxei akìmh k^poia oloklhrwmènh lÔsh sto par^doxo, ìmwc up^rqoun di^forec idèec.

Sto pr¸to mèroc thc ergasÐac aut c parousi^zontai oi basikèc ènnoiec pou qrhsi-mopoioÔntai sthn hmiklasik melèth twn barutik¸n susthm^twn kai dÐnetai idiaÐterh shmasÐa sthn efarmog touc stic maÔrec trÔpec. EpÐshc parousi^zontai ènnoiec pou sqetÐzontai me thn plhroforÐa pou metafèrei èna sÔsthma kaj¸c kai ènnoiec pou sqetÐzontai me thn kban-tik hlektrodunamik (QED). Sto deÔtero mèroc arqik^ parousi^zontai exeidikeumènec ènnoiec pou sqetÐzontai me thn melèth miac apì tic epikratèsterec idèec gia thn lÔsh tou pio p^nw paradìxou. AkoloÔjwc parousi^zontai kai genikeÔontai posotikoÐ upologismoÐ pou enisqÔoun aut thn idèa, upologismoÐ oi opoÐoi basÐzontai se prohgoÔmenh ergasÐa [1].

Mèroc I

MaÔrec trÔpec, par^doxa kai IR fainìmena

Kef^laio 1

MaÔrec trÔpec

1.1 LÔsh Schwarzschild

H lÔsh Schwarzschild apoteleÐ lÔsh twn exis¸sewn Eistein gia thn exwterik perioq sfairik^ summetrik¸n phg¸n (ìpwc ènac sfairik^ summetrikìc mh peristrefìmenoc astèrac). H metrik sthn perÐptwsh aut èqei thn morf

$$dS^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.1.1}$$

ìpou M eÐnai h m^za-enèrgeia thc phg c kai $c = 1$. ParathroÔme ìti se meg^lec apost^seic apì thn phg , h metrik teÐnei asumptwtik^ sthn metrik Minkowski ìpwc anamèname. EpÐshc gia M=0 paÐrnoume thn metrik Minkowski afoÔ den up^rqei phg gia na prokalèsei kampÔlwsh tou q¸rou. An h m^za eÐnai katanemhmènh se aktÐna mikrìterh apì 2MG tìte h metrik Schwarzschild parousi^zei apeirismì gia $r = 2MG$ kai èqoume thn dhmiourgÐa mia melan c op c. H aktÐna $r = 2MG$ onom^zetai aktÐna Schwarzschild barutik aktÐna kai h epif^neia pou kajorÐzetai apì aut thn aktÐna onom^zetai orÐzontac. EpÐshc h metrik apeirÐzetai kai gia $r = 0$.

UpologÐzontac to ginìmeno tou tanust Riemann me ton eautì tou brÐskoume ìti

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \tag{1.1.2}$$

ParathroÔme ìti h kampulìthta gia $r = 2MG$ eÐnai peperasmènh, kai epomènwc den apoteleÐ pragmatik anwmalÐa.
Oi suntetagmènec pou qrhsimopoi same den mporoÔn na perigr^youn thn perioq kont^ ston orÐzonta. Gia $r = 0$ h kampulìthta apeirÐzetai, kai epomènwc aut eÐnai mia pragmatik anwmalÐa, thn opoÐa den mporoÔme na apaleÐyoume me k^poio metasqhmatismì suntetagmènwn.

Kaj¸c plhsi^zoume ton orÐzonta h metabol tou qrìnou Schwarzschild dt apeirÐzetai (gia stajerèc metabolèc tou idiìqronou) kaj¸c o suntelest c tou ìrou dt^2 mhdenÐzetai. Epomènwc ènac parathrht c pou brÐsketai se meg^lh apìstash apì thn maÔrh trÔpa ja blèpei k^poion parathrht pou pèftei eleÔjera na epibradÔnetai kai na plhsi^zei asumptwtik^ ton orÐzonta qwrÐc na ton diapern^ potè.

E^n upologÐsoume ton idiìqrono pou qrei^zetai ènac parathrht c pou pèftei eleÔjera gia na per^sei ton orÐzonta (xekin¸ntac apì th jèsh $r = r_0$) brÐskoume

$$\tau = \frac{\pi r_0}{2}\left(\frac{r_0}{2MG}\right) \tag{1.1.3}$$

O parathrht c ft^nei sthn anwmalÐa $r = 0$ se peperasmèno idiìqrono. AfoÔ di^foroi parathrhtèc katal goun sto shmeÐo $r = 0$, autì apoteleÐ fusikì shmeÐo kai prèpei na èqei peperasmènec fusikèc posìthtec. Epomènwc gia $r \to 0$ oi exis¸seic tou Eistein apotugq^noun na apod¸soun swst^ th fusik sumperifor^.

Kaj¸c pern^me ton orÐzonta oi suntetagmènec t kai r all^zoun eÐdoc. Gia $r < 2MG$ h sunist¸sa r gÐnetai qronik kai h sunist¸sa t qwrik epomènwc opoiod pote s¸ma fwc per^sei to orÐzonta katal gei sthn anwmalÐa $r = 0$. H anwmalÐa den apoteleÐ shmeÐo tou q¸rou pou ja mporoÔse na to apofÔgei kaneÐc all^ èna mellontikì gegonìc.

1.1.1 Suntetagmènec tortoise

Jèloume na broÔme kat^llhlec suntetagmènec gia na perigr^youme thn exwterik perioq miac maÔrhc trÔpac. 'Enac trìpoc na to k^noume autì eÐnai orÐzontac thn aktinik suntetagmènh tortoise r^* [2]

$$\frac{1}{1-\frac{2MG}{r}}dr^2 = \left(1-\frac{2MG}{r}\right)(dr^*)^2 \tag{1.1.4}$$

H metrik paÐrnei thn morf

$$ds^2 = -\left(1-\frac{2MG}{r}\right)\left[dt^2 - (dr^*)^2\right] + r^2 d\Omega^2 \tag{1.1.5}$$

H suntetagmènh tortoise sqetÐzetai me thn suntetagmènh r me thn akìlouja sqèsh

$$r^* = r + 2MG\ln\left(\frac{r-2MG}{2MG}\right) \tag{1.1.6}$$

kai paÐrnei timèc se olìklhrh thn eujeÐa, $r* \in (-\infty, \infty)$. Blèpoume ìti h metrik sthn exwterik perioq miac maÔrhc trÔpac paÐrnei mia apl morf stic suntetagmènec tortoise. Sugkekrimèna h metrik stic disdi^statec fètec $\theta, \varphi =$staj eÐnai sÔmmorfh epÐpedh.

1.1.2 Suntetagmènec Rindler

OrÐzoume thn suntetagmènh ρ na eÐnai h fusik apìstash enìc shmeÐou apì ton orÐzonta

$$\rho = \int_{2MG}^{r} dr' \left(1 - \frac{2MG}{r'}\right)^{-\frac{1}{2}} \tag{1.1.7}$$

H metrik paÐrnei thn morf

$$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + d\rho^2 + r^2 d\Omega^2 \tag{1.1.8}$$

Jèloume na melet soume thn metrik polÔ kont^ ston orÐzonta. Gia $r \to 2MG$

$$\rho \approx 2\sqrt{2MG(r - 2MG)} \tag{1.1.9}$$

kai

$$ds^2 = -\rho^2 d\omega^2 + d\rho^2 + r^2 d\Omega^2, \quad \omega = \frac{t}{4MG} \tag{1.1.10}$$

Jètoume

$$X = 2MG \sin\theta \cos\varphi, \quad Y = 2MG \sin\theta \sin\varphi, \quad T = \rho \sinh\omega, \quad Z = \rho \cosh\omega \tag{1.1.11}$$

kai esti^zoume se mia arket^ mikr perioq gÔrw apì to bìreio pìlo thc sfaÐrac, $\theta = 0$. Tìte paÐrnoume thn metrik Minkowski

$$ds^2 = -dT^2 + dZ^2 + dX^2 + dY^2 \tag{1.1.12}$$

H prosèggish aut eÐnai kal gia mikrèc apost^seic apì ton orÐzonta kai gia mikrèc gwniakèc metabolèc. DeÐqnei ìti den up^rqei kami^ anwmalÐa ston orÐzonta.

1.1.3 Suntetagmènec Kruskal-Szekeres

Se autì to kef^laio ja prospaj soume na ekfr^soume thn metrik Schwarzschild se morf pou na eÐnai sÔmmorfh me thn metrik Minkowski. 'Otan lème ìti h metrik eÐnai sÔmmorfh me mia ^llh metrik ennooÔme ìti diafèroun mìno pollaplasiastik^. 'Etsi autèc oi metrikèc èqoun di^forec koinèc idiìthtec pou kajist^ thn melèth touc pio apl . Sthn perÐptwsh pou mia metrik eÐnai sÔmmorfh epÐpedh, oi fwtoeid c gewdaitikèc eÐnai eujeÐec me klÐsh ± 1. EpÐshc oi sÔmmorfoi metasqhmatismoÐ diathroÔn thn sqèsh aitiìthtac metaxÔ dÔo opoiwnd pote geitonik¸n shmeÐwn.

Gr^foume loipìn thn metrik Schwarzschild sthn morf :

$$ds^2 = F(R)[-R^2 d\omega^2 + dR^2] - r^2 d\Omega^2 \tag{1.1.13}$$

H qronik suntetagmènh ω èqei oriste□ prohgoumènwc. H suntetagmènh R kai h sunˆrthsh F pa□rnoun thn morf

$$R = MGe^{\frac{r}{4MG}}\sqrt{\frac{r}{2MG} - 1} \tag{1.1.14}$$

kai

$$F = \frac{32MG}{r}e^{-\frac{r}{2MG}} \tag{1.1.15}$$

¸ste to apeirostì diˆsthma na paramènei analllo□wto.

AkoloÔjwc or□zoume 2 fwtoeid c suntetagmènec

$$U = -Re^{-\omega}, \qquad V = Re^{\omega} \tag{1.1.16}$$

kai h epagìmenh metrik se mia fèta $\theta, \varphi =$staj pa□rnei thn morf

$$dS^2 = -F(R)dUdV \tag{1.1.17}$$

Oi suntetagmènec Kruskal-Szekeres or□zontai wc

$$T = \frac{V+U}{2}, \qquad Z = \frac{V-U}{2} \tag{1.1.18}$$

kai h metrik se mia aktinik fèta pa□rnei th morf

$$dS^2 = F(R)[-dT^2 + dZ^2] \tag{1.1.19}$$

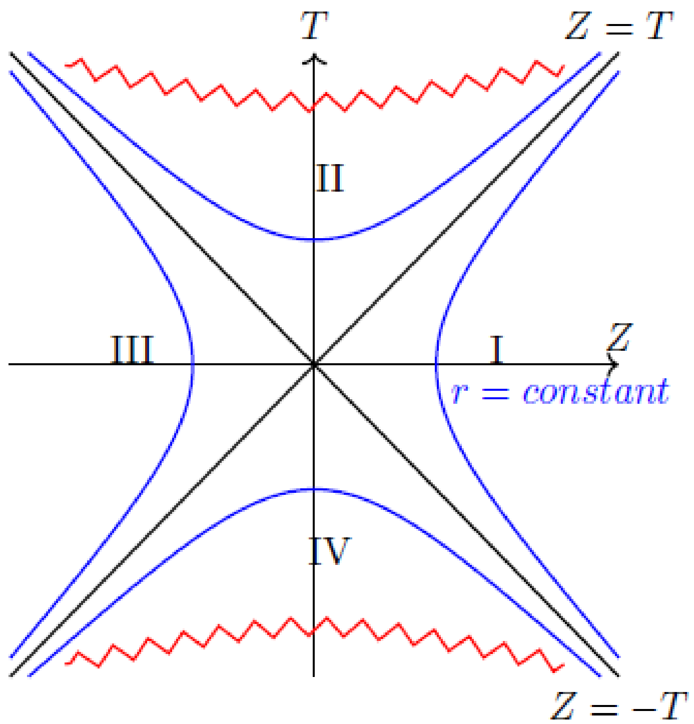
Epekte□noume analutikˆ tic suntetagmènec T,Z se ìlo to ep□pedo kai to diˆgramma thc mègisth gewmetr□a Schwarzschild pa□rnei thn morf thc eikìnac 1.1.1.

1.2 Diagrˆmmata Penrose

Ta diagrˆmmata Penrose apoteloÔn èna apì ta pio qr sima ergale□a sthn katanìhsh gewmetri¸n sthn genik jewr□a thc sqetikìthtac. H idèa e□nai na apeikon□soume ton qwrìqrono me èna peperasmèno diˆgramma, ektel¸ntac èna sÔmmorfo metasqhmatismì suntetagmènwn.

Ta diˆfora ˆpeira sto arqikì diˆgramma metasqhmat□zontai se peperasmèna shme□a sto diˆgramma Penrose wc ex c

$$r = +\infty,\ t = \text{staj} \to i^0, \quad t = +\infty,\ r = \text{staj} \to i^+, \quad t = -\infty,\ r = \text{staj} \to i^- \tag{1.2.1}$$



EIKONA 1.1.1: Mègisth gewmetrÐa Schwarzschild stic suntetagmènec Kruskal-Szekeres. H maÔrh diag¸nia eujeÐa me klÐsh $+1$ eÐnai o mellontikìc orÐzontac gegonìtwn. H anwmalÐa $r = 0$ anaparÐstatai apì thn kìkkinh gramm . H perioq II eÐnai melan eeid den mporeÐ na diafÔgei apì aut n plhroforÐa. H perioq IV eÐnai mia leuk op sthn opoÐa den mporeÐ na eisèljei plhroforÐa. Leukèc opèc den èqoun parathrhjeÐ sthn fÔsh (parabi^zoun ton deÔtero jermodunamikì nìmo k^ti pou den apagoreÔetai apìluta all^ eÐnai polÔ apÐjano na sumbeÐ).

Ta fwtoeid ^peira, dhlad ta shmeÐa sta opoÐa arqÐzoun kai katal goun oi fwtoeideÐc gewdaitikèc, sumbolÐzontai me $\mathcal{I}^+$ gia to mellontikì fwtoeidèc ^peiro kai $\mathcal{I}^-$ gia to parelqontikì fwtoeidèc ^peiro. EpÐshc, ta diagr^mmata Penrose anadeiknÔoun tic aitiatikèc sqèseic metaxÔ dÔo opoiwnd pote qwroqronik¸n shmeÐwn, oi opoÐec den eÐnai aparaÐthta profaneÐc sthn arqik apeikìnish.

1.2.1 Qwrìqronoc Minkowski

Se sfairikèc suntetagmènec h metrik Minkowski dÐnetai apì thn èkfrash

$$dS^2 = -dt^2 + dr^2 + r^2 d\Omega_2 \tag{1.2.2}$$

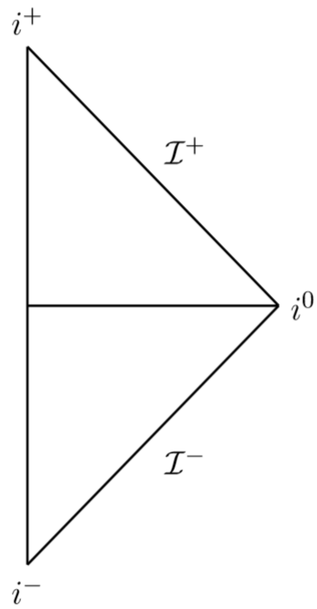
Ja melet soume mia disdi^stath fèta ìpou $\theta, \varphi =$staj. OrÐzoume tic suntetagmènec

$$U = \tanh(t - r), \quad V = \tanh(t + r) \tag{1.2.3}$$

kai ètsi ta ^peira gÐnontai

$$i^+ : U = V = 1, \quad i^0 : U = -V = -1, \quad i^- : Y - V = -1 \tag{1.2.4}$$

A foÔ broÔme aut^ ta shmeÐa mporoÔme na sqedi^soume to di^gramma Penrose.



Eikona 1.2.1: Di^gramma Penrose gia thn metrik Minkowski

1.2.2 Qwrìqronoc Schwarzschild

Ja qrhsimopoi soume tic fwtoeideÐc suntetagmènec Kruskal-Szekeres pou orÐzontai wc:

$$U = -Re^{-\omega}, \quad V = Re^{\omega} \tag{1.2.5}$$

ìpou $\omega = t/4GM$, $R = GM \exp\{(r^*/4GM)\}$ kai $r^* = r + 2GM \ln(r/2GM - 1)$ (gia $r > 2GM$). OrÐzoume tic suntetagmènec T, Z wc akoloÔjwc

$$T = \frac{U+V}{2}, \quad Z = \frac{V-U}{2} \tag{1.2.6}$$

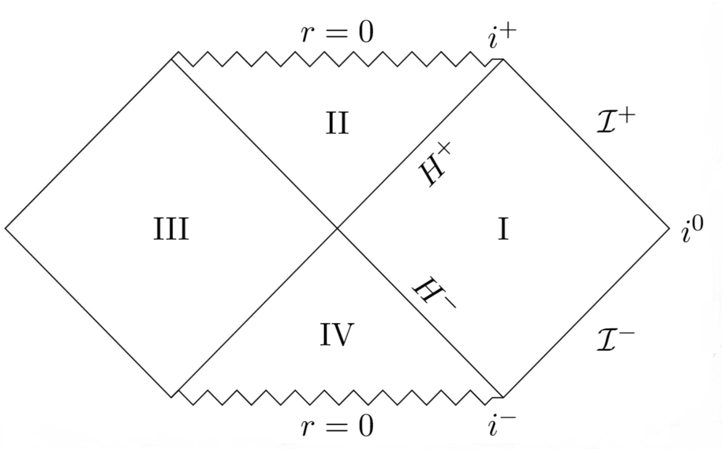
oi opoÐec epikalÔptoun olìklhro ton qwrìqrono. Gia na sqedi^soume to di^gramma Penrose orÐzoume tic suntetagmènec

$$\bar{U} = \tanh(T - Z), \quad \bar{V} = \tanh(T + Z) \tag{1.2.7}$$

Tèloc orÐzoume mia nèa qronik kai mia nèa qwrik suntetagmènh

$$\bar{T} = \frac{\bar{U}+\bar{V}}{2}, \quad \bar{Z} = \frac{\bar{V}-\bar{U}}{2} \tag{1.2.8}$$

BrÐskontac ta di^fora ^peira ìpwc sthn perÐptwsh Minkowski kataskeu^zoume ton di^gramma Penrose gia thn megÐsth gewmetrÐa Schwarzschild ìpou H^+ kai H^- eÐnai o mel-



Eikona 1.2.2: Di^gramma Penrose gia thn megÐsth gewmetrÐa Schwarzschild

lontikìc kai parel jontikìc orÐzontac gegonìtwn antÐstoiqa.

1.3 Metrik Vaidya

'Opwc eÐdame h metrik Schwarzschild epitrèpei thn Ôparxh perioq¸n sto qwrìqrono apì tic opoÐec den mporeÐ na diafÔgei plhroforÐa, perioqèc tic opoÐec onom^same melanèc. 'Omwc h mègisth gewmetrÐa Schwarzschild epitrèpei kai thn Ôparxh leuk¸n op¸n, oi opoÐec den èqoun parathrhjeÐ sthn fÔsh. Gia na peisjoÔme gia thn Ôparxh twn melan¸n op¸n ja melet soume se autì to kef^laio thn dhmiourgÐa enìc tètoiou antikeimènou mèsw thc diadikasÐac thc barutik c kat^rreushc.

H aploÔsterh perÐptwsh eÐnai na èqoume èna sfairikì floiì fwtìc apeiros toÔ p^qouc, o opoÐoc katarrèei aktinik^ [2, 3, 4]. Me b^sh to je¸rhma tou Birkhoft [5], h gewmetrÐa sthn exwterik perioq miac sfairik^ summetrik c pug c eÐnai h gewmetrÐa Schwarzschild, akìmh kai sthn perÐptwsh pou h phg den eÐnai statik . Epomènwc h metrik sthn exwterik perioq tou floioÔ mporeÐ na graf teÐ sthn morf

$$dS^2 = -\left(1 - \frac{2m(v)}{r}\right) dv^2 + 2drdv + r^2 d\Omega^2 \tag{1.3.1}$$

ìpou

$$v = t + r + 2Gm(v)\ln\left|\frac{r}{2m(v)G} - 1\right| \tag{1.3.2}$$

kai

$$m(v) = M \quad \text{gia} \quad v > v_0 \tag{1.3.3}$$

$$m(v) = 0 \quad \text{gia} \quad v < v_0 \tag{1.3.4}$$

ìpou M h olik enèrgeia tou f loioÔ. Sthn eswterik perioq tou f loioÔ èqoume $m(v) = 0$, epomènwc h metrik eÐnai h epÐpedh metrik Minkowski. H troqi^ tou fwtìc perigr^fetai apì thn exÐswsh $v = v_0$.

Qrhsimopoi¸ntac thn pio p^nw metrik brÐskoume ìti h mình mh mhdenik sunist¸sa tou tanust Ricci eÐnai h

$$R_{vv} = \frac{2\mathcal{E}(v)}{r^2} \tag{1.3.5}$$

ìpou $\mathcal{E}(v) = M\delta(v - v_0)$ kai h bajmwt kampulìthta Ricci mhdenÐzetai. 'Etsi qrhsimopoi¸ntac tic exis¸seic tou Eistein brÐskoume ton tanust enèrgeiac-orm c

$$T_{vv} = \frac{M\delta(v - v_0)}{4\pi r^2} \tag{1.3.6}$$

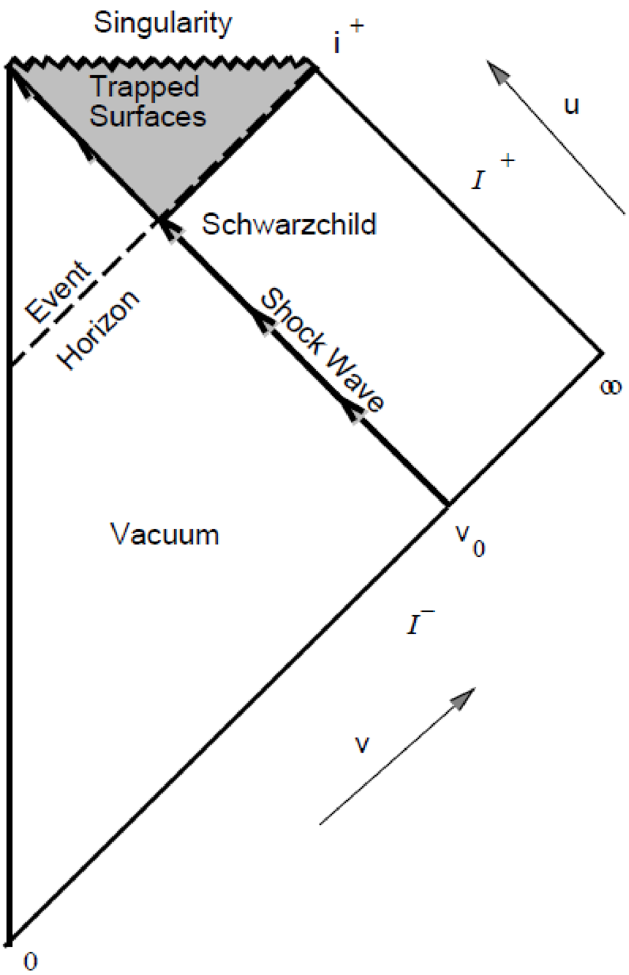
me tic upìloipec sunist¸sec tou Ðsec me 0.

Gia meg^la r, h suntetagmènh v paÐrnei th morf $v = t + r$, kai h metrik teÐnei asumptwtik^ sthn epÐpedh metrik Mnkwowski. O tanust c enèrgeiac-orm c stic suntetagmènec t, r paÐrnei thn morf :

$$T_{tt} = T_{tr} = T_{rr} = T_{vv} \tag{1.3.7}$$

ìpwc anamèname. H puknìthta enèrgeiac (T_{tt}) tou sfairikoÔ f loioÔ fwtìc prèpei na isoÔtai me thn puknìthta thc orm c sthn aktinik kateÔjunsh (T_{tr}), afoÔ èqoume kajar^ aktinik kÐnhsh. Oi pio p^nw puknìthtec isoÔntai me to reÔma thc aktinik c sunist¸sac thc orm c (T_{rr}). Upenjum Ðzoume ìti qrhsimopoioÔme mon^dec ìpou $c = 1$.

Gia na oloklhrwjeÐ h lÔsh tou probl matoc prèpei na sundèsoume tic lÔseic gia thn eswterik kai exwterik perioq tou sfairikoÔ kelÔfouc. Gia na broÔme tic kat^llhlec sunoriakèc sunj kec, oloklhr¸noume tic exis¸seic Einstein diamèsou tou f loioÔ. ProkÔptei ìti h metrik eÐnai suneq c diamèsou tou f loioÔ kai oi pr¸tec par^gwgoi thc metrik c parousi^zoun asunèqeia b matoc. Gia na diasfalÐsoume th sunèqeia sthn metrik epilègoume thn monadik eleÔjerh par^metro tou probl matoc v_0 ètsi ¸ste ta 'kèntra' $r = 0$ thc eswterik c kai thc exwterik c lushc na sumpÐptoun. To di^gramma Penrose apeikonÐzetai sthn eikìna 1.3.1.



Eikona 1.3.1: Diˆgramma Penrose gia thn metrik Vaidya[2]. To fwc kineÐtai aktinikˆ proc to kèntro tou sfairikoÔ floioÔ mèqri na eisèljei entìc thc aktÐnac Schwarzschild, opìte kai dhmiourgeÐtai h melan op . AkoloÔjwc suneqÐzei thn aktinik tou poreÐa mèqri na katal xei sthn anwmalÐa.

1.4 Fortismènec maÔrec trÔpec

'Eqoume mia mh peristrefìmenh, fortismènh kai sfairikˆ summetrik maÔrh trÔpa. JewroÔme ìti h metrik sthn exwterik perioq thc maÔrhc trÔpac èqei thn genik morf [2]

$$dS^2 = -e^{2a(r,t)}dt^2 + e^{2b(r,t)}dr^2 + r^2 d\Omega^2 \tag{1.4.1}$$

ExaitÐac thc sfairik c summetrÐac h mình mh mhdenik sunist¸sa tou hlektrikoÔ pedÐou eÐnai h aktinik :

$$E_r = f(r,t) \tag{1.4.2}$$

Tìte o tanust c enèrgeiac-orm c tou hlektromagnhtikoÔ pedÐou, sthn parousÐa barÔthtac, dÐdetai apì thn èkfrash

$$T_{\mu\nu} = F_{\mu\rho}F^{\rho}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{1.4.3}$$

Gia na broÔme thn metrik lÔnoume tic exis¸seic Einstein

$$R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu} \tag{1.4.4}$$

To Ðqnoc tou tanust enèrgeiac-orm c eÐnai 0, epomènwc katal goume stic exis¸seic

$$R_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.4.5}$$

Qrhsimopoi¸ntac autèc tic exis¸seic kai tic exis¸seic Maxwell sto kenì

$$F^{\alpha\beta};\alpha = 0, \quad \frac{\partial F^{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial F^{\gamma\alpha}}{\partial x^{\beta}} + \frac{\partial F^{\alpha\beta}}{\partial x^{\gamma}} = 0 \tag{1.4.6}$$

brÐskoume thn metrik Reissner-Nordstrom

$$dS^2 = -\left(1 - \frac{2GM}{r} + \frac{Q^2G}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2G}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{1.4.7}$$

To hlektrikì pedÐo eÐnai

$$E_r = \frac{Q}{r^2} \tag{1.4.8}$$

E˜n to hlektrikì pedÐo sthn epif˜neia thc maÔrhc trÔpac eÐnai polÔ meg˜lo, tìte par˜gontai zeÔgh hlektronÐwn-pozitronÐwn. E˜n h maÔrh trÔpa eÐnai jetik˜ fortismènh, ja èlkei ta hlektrìnia kai ja apwjeÐ ta pozitrìnia. Ta hlektrìnia ja arqÐsoun na eisèrqontai sthn maÔrh trÔpa, me apotèlesma na mei¸netai stadiak˜ to fortÐo thc maÔrhc trÔpac, mèqri na exafanisteÐ entel¸c (fainìmeno Schwinger). Gia na to apofÔgoume to fainìmeno autì, epib˜loume h tim tou hlektrikoÔ pedÐou sthn epif˜neia thc maÔrhc trÔpac na eÐnai polÔ mikrìterh apì thn m˜za tou hlektronÐou, ¸ste na mhn mporoÔn eÔkola na paraqjoÔn zeÔgh hlektronÐwn-pozitronÐwn

$$\frac{M^2}{Q} << m_e^2 \tag{1.4.9}$$

me $r \sim GM$ ston orÐzonta.

Jètontac $g_{00} = 0$ brÐskoume ìti up˜rqoun dÔo orÐzontec

$$r_{\pm} = MG\left(1 \pm \sqrt{1 - \frac{Q^2}{M^2G}}\right) \tag{1.4.10}$$

Eˆn $Q^2 > M^2G$ oi lÔseic eÐnai migadikèc kai mh apodektèc. Sthn perÐptwsh aut h anwmalÐa $r = 0$ den kalÔptetai apì orÐzonta (gumn anwmalÐa). Autˆ ta s¸mata den mporoÔn na perigrafoÔn me bˆsh thn genik sqetikìthta.
Sthn perÐptwsh ìpou $M^2 = \frac{Q^2}{G}$, oi dÔo orÐzontec sumpÐptoun kai h maÔrh trÔpa onomˆzetai oriak (extremal).

AkoloÔjwc ja exetˆsoume thn perÐptwsh ìpou $Q^2 < M^2G$. Jètoume

$$r_+ - r_- \equiv \Delta, \quad r_+ + r_- \equiv \Sigma, \quad r - \frac{\Sigma}{2} \equiv y \tag{1.4.11}$$

'Etsi h metrik kontˆ ston orÐzonta paÐrnei thn morf

$$dS^2 = -\frac{\Delta^2}{4r_+^4}\rho^2 dt^2 + d\rho^2 = -\rho^2 d\omega^2 + d\rho^2, \quad \omega \equiv \frac{\Delta}{2r_+}t \tag{1.4.12}$$

gia $\theta, \varphi =$ stajerˆ kai ρ h fusik apìstash apì ton orÐzonta.

H epifaneiak puknìthta fortÐou eÐnai $Q/4\pi r_+^2$, kai afoÔ $r_+ \sim GM$, h puknìthta gÐnetai anˆlogh tou $Q/4\pi M^2G^2$. ParathroÔme ìti ìso auxˆnetai h mˆza tìso elatt¸netai h puknìthta fortÐou. Epomènwc gia megˆlec maÔrec trÔpec h èntash tou hlektrikoÔ pedÐou kontˆ ston orÐzonta eÐnai amelhtèa. SumperaÐnoume ìti ta qarakthristikˆ thc fortismènhc maÔrhc trÔpac eÐnai parìmoia me autˆ thc maÔrhc trÔpac Schwarzschild. H energ jermokrasÐa kontˆ ston orÐzonta dÐnetai apì thn èkfrash

$$T_{eff} = \frac{1}{2\pi\bar{\rho}} \tag{1.4.13}$$

Asumptwtikˆ, se megˆlec apostˆseic h jermokrasÐa paÐrnei thn tim

$$T_\infty = \frac{\Delta}{4\pi r_+^2} = \frac{2GM\left(1 - \frac{Q^2}{M^2G}\right)^{\frac{1}{2}}}{4\pi M^2G^2\left(1 + \sqrt{1 - \frac{Q^2}{M^2G}}\right)^2} \tag{1.4.14}$$

Sto ìrio ìpou h maÔrh trÔpa gÐnetai oriak $(\Delta \to 0)$, h apìstash r dÐnetai apì thn èkfrash

$$\rho \to y + \frac{\Sigma}{2}\left(\log(2y) - \log(\Delta)\right) \tag{1.4.15}$$

kai h metrik paÐrnei thn morf :

$$dS^2 = -\left[r + \sinh\left(\frac{\rho}{r_+}\right) d\omega^2 + d\rho^2\right] \tag{1.4.16}$$

Gia $\rho/r_+ \ll 1$ h metrik proseggÐzetai apì thn metrik Rindler. EpÐshc oi oriakèc maÔrec trÔpec èqoun asumptwtikˆ mhdenik jermokrasÐa kai ètsi h enèrgeia touc den mei¸netai. Oi oriakèc maÔrec trÔpec apoteloÔn th basik katˆstash twn fortismènwn maÔrwn trup¸n.

H m^za miac fortismènhc maÔrhc trÔpac mei¸netai mèqri na p^rei thn el^qisth oriak thc tim . Oi oriakèc maÔrec trÔpec exaul¸nontai mèsw ^llwn diergasi¸n ìpwc to fainìmeno Schwinger. An kai èqoun mhdenik jermokrasÐa, h entropÐa touc eÐnai mh mhdenik epeid èqoun orÐzonta. H entropÐa eÐnai Ðsh me [1]

$$S = \frac{A}{4G} = \frac{\pi r_+^2}{G} \tag{1.4.17}$$

Sthn oriak perÐptwsh h entropÐa isoÔtai me $S = \pi M^2 G$.

[1] Ο γενικός τύπος για την εντροπία μιας μαύρης τρύπας αποδεικνύεται στο επόμενο κεφάλαιο.

Kef ˆ laio 2

Jermodunamik pedÐwn se kampulwmènouc qwr ì qronouc

2.1 PedÐa se kampulwmènouc qwr ì qronouc

2.1.1 Klassik ˆ bajmwt ˆ pedÐa sthn gewmetrÐa Schwarzschild

Jèloume na melet soume pwc sumperifèretai èna klassikì bajmwtì pedÐo ston qwrìqrono Schwarzschild. Gia na to k ˆ noume autì ja qrhsimopoi soume thn metrik tortoise

$$dS^2 = F(r^*)[-dt^2 + dr^{*2}] + r^2 d\Omega^2 \tag{2.1.1}$$

H metrik aut eÐnai sÔmmorfh me thn metrik Minkowski.

'Estw ìti èqoume èna eleÔjero, ˆ mazo pedÐo Klein-Gordon X. H dr ˆ sh tou pedÐou orÐzetai wc

$$dS^2 = \int d^4x \sqrt{-g}\, L\ (\varphi, \partial_\mu \varphi) \tag{2.1.2}$$

ìpou

$$L\ (\varphi, \partial_\mu \varphi) = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \tag{2.1.3}$$

Sthn perÐptwsh tou epÐpedou qwrìqronou Minkowski, h dr ˆ sh ja tan h pio p ˆ nw me $g = \det(g_{\mu\nu}) = -1$. SÔmfwna me thn arq thc genik c sqetikìthtac, mia exÐswsh pou isqÔei sthn apousÐa barÔthtac kai paramènei amet ˆ blhth k ˆ tw apì genikoÔc metasqhmatismoÔc suntetagmènwn isqÔei kai sthn parousÐa barÔthtac[1].

[1] Η δράση είναι αμετάβλητη κάτω από γενικούς μετασχηματισμούς συντεταγμένων, αφού η Λαγκραντζιανή πυκνότητα αποτελεί εσωτερικό γινόμενο τανυστών και ο στοιχειώδης όγκος $d^4x\sqrt{-g}$ είναι αναλλοίωτος.

OrÐzoume to pedÐo $\Psi = r\varphi$ to opoÐo gr^foume wc seir^ sfairik¸n armonik¸n (epeid h metrik Schwarzschild eÐnai sfairik^ summetrik) kai katal goume sthn exÐswsh

$$dS^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -\frac{1}{2} \int dt \int dr^* \left([-(\partial_t \Psi_{lm})^2 + (\partial_{r^*} \Psi lm)^2 + V_l(r^*) \left| \Psi_{lm} \right| \right) \tag{2.1.4}$$

$$V_l(r^*) = \left(\frac{r - 2GM}{r} \right) \left(\frac{l(l+1)}{r^2} + \frac{2GM}{r^3} \right) \tag{2.1.5}$$

To V_l eÐnai to energì dunamikì.

O pr¸toc ìroc tou V_l eÐnai o fugìkentroc ìroc (apwstikìc ìroc) kai o deÔteroc ìroc eÐnai to dunamikì Schwarzschild. To dunamikì paÐrnei thn mègisth tim tou sto shmeÐo

$$r_0 = 3MG \left(\frac{1}{2} \left[1 + \sqrt{1 + \frac{14l^2 + 14l + 9}{9l^2(l+1)^2}} \right] - \frac{1}{2l(l+1)} \right) \tag{2.1.6}$$

Se apost^seic makri^ apì to orÐzonta (gia $r > r_0$) to dunamikì eÐnai apwstikì, en¸ kont^ ston orÐzonta (gia $r < r_0$) to dunamikì eÐnai elktikì. Gia $l \to \infty$ èqoume $r_0 = 3GM$ en¸ gia $l = 0$ èqoume $r_0 = 8GM/3$. To mègisto tou dunamikoÔ paÐrnei el^qisth tim gia $l = 0$ epomènwc ta s-kÔmata pou par^gontai lÐgo èxw apì ton orÐzonta èqoun thn megalÔterh pijanìthta na dièljoun tou fr^gmatoc, kai suneisfèroun entonìtera sthn aktinobolÐa Hawking, ìpwc ja doÔme argìtera. To mègisto tou dunamikoÔ sthn perÐptwsh $l = 0$ eÐnai

$$V_0 = \frac{1}{2M^2G^2} \left(\frac{3}{8} \right)^3 \tag{2.1.7}$$

en¸ gia meg^la l paÐrnoume

$$V_0 \simeq \frac{l^2}{27M^2G^2} \tag{2.1.8}$$

Gia na broÔme thn exÐswsh kÐnhshc paÐrnoume thn Lagkrantzian apì thn dr^sh

$$L = -\frac{1}{2} \left(-(\partial_t \Psi_{lm})^2 + (\partial_{r^*} \psi_{lm})^2 + V_l(r^*) \left| \Psi_{lm} \right|^2 \right) \tag{2.1.9}$$

kai qrhsimopoi¸ntac tic exis¸seic Euler-Lagrange brÐskoume

$$\partial_t^2 \Psi_{lm} = \partial_{r^*}^2 \Psi_{lm} - V_l(r^*) \Psi_{lm} \tag{2.1.10}$$

Gia na lÔsoume thn exÐswsh kÐnhshc dokim^zoume qwrizìmenh lÔsh thc morf c

$$\Psi_{lm}(t, r^*) = e^{-irt} \Psi_{lm}(r^*) \tag{2.1.11}$$

Oi lÔseic me jetik c suqnìthta perigr^foun swmatÐdia arnhtik c enèrgeiac ta opoÐa den mporoÔn na dièljoun apì to fr^gma dunamikoÔ. Sundu^zontac tic exis¸seic 2.1.10 kai 2.1.11 paÐrnoume

$$r^2\Psi_{lm}(r^*) = -\frac{d^2\Psi_{lm}(r^*)}{dr^{*2}} + V_l(r^*)\Psi_{lm}(r^*) \tag{2.1.12}$$

pou èqei thn morf thc monodi^stathc qronoanex^rththc exÐswshc Schrodinger, gia èna swmatÐdio me enèrgeia v^2. Epomènwc blèpoume ìti swmatÐdia me $l = 0$ prèpei na èqoun enèrgeia

$$v = \frac{1}{\sqrt{2}GM}\left(\frac{3}{8}\right)^{\frac{3}{2}} \tag{2.1.13}$$

gia na diafÔgoun apì to fr^gma dunamikoÔ (me b^sh to fainìmeno s raggac diafeÔgei amelhtèoc arijmìc swmatidÐwn). Gia swmatÐdia me meg^lo l (meg^lh stroform)

$$v \simeq \frac{l}{\sqrt{27}GM}\left(\frac{3}{8}\right)^{\frac{3}{2}} \tag{2.1.14}$$

pou eÐnai polÔ megalÔterh apì thn perÐptwsh me $l = 0$.

Kont^ ston orÐzonta to dunamikì mporeÐ na jewrhjeÐ amelhtèo epomènwc èqoume

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}}\right)\Psi_{lm} = 0 \tag{2.1.15}$$

pou eÐnai h kumatik exÐswsh. Oi lÔseic paÐrnoun thn morf epÐpedwn kum^twn

$$\Psi_{lm}(t, r^*, \theta, \varphi) = e^{ik(r^* \pm t)}\Psi_{lm}(\theta, \varphi) \tag{2.1.16}$$

2.1.2 Klasik^ bajmwt^ pedÐa sthn gewmetrÐa Ridler

Kont^ ston orÐzonta miac maÔrhc trÔpac mporoÔme na perigr^youme thn gewmetrÐa me thn metrik Ridler. H metrik Ridler gr^fetai sthn morf

$$dS^2 = -\rho^2 d\omega^2 + d\rho^2 + dX^2 + dY^2 \tag{2.1.17}$$

K^noume thn allag suntetagmènwn

$$u = \log\rho \tag{2.1.18}$$

kai h metrik paÐrnei thn morf

$$dS^2 = e^{2u}(-d\omega^2 + du^2) + dX^2 + dY^2 \tag{2.1.19}$$

H suntetagmènh u paÐrnei timèc se olìklhrh thn eujeÐa, $u \in \{-\infty, \infty\}$, kai h nèa metrik c eÐnai sÔmmorfh me thn metrik Minkowski.

H dr^sh gia èna ^mazo bajmwtì pedÐo φ eÐnai

$$dS^2 = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = -\frac{1}{2}\int d^4xe^{2u}\left(e^{-2u}\left[-(\partial_\omega\varphi)^2 + (\partial_u\varphi)^2\right] + (\partial_\perp\varphi)^2\right) \tag{2.1.20}$$

ìpou $\partial_\perp = (\partial_X, \partial_Y)$. Epeid oi suntetagmènec Q,Y eÐnai epÐpedec, mporoÔme na qrhsimopoi soume metasqhmatismoÔc Fourier stic dÔo dias t^seic gia to pedÐo φ

$$\varphi = \int d^2k_\perp e^{ik_\perp x_\perp} x(k_\perp, \omega, u) \tag{2.1.21}$$

AnalÔsame to pedÐo wc epallhlÐa epÐpedwn kum^twn. H dr^sh paÐrnei thn morf

$$S = -\frac{1}{2}\int dud\omega\left[-(\partial_\omega x)^2 + (\partial_u x)^2 + k^2e^{2u}x^2\right] \tag{2.1.22}$$

OrÐzoume wc energì m^za (dunamikì) thn posìthta

$$m^2_{eff}(k,u) = k^2e^{2u} = V \tag{2.1.23}$$

Kaj¸c to $u \to -\infty$ ($\rho \to 0$) to dunamikì mhdenÐzetai gia ìlouc touc tÔpouc tal^ntwshc. Kaj¸c apomakrunìmaste apì ton orÐzonta to dunamikì aux^netai ekjetik^, kai mìno sthn perÐptwsh $k = 0$ to dunamikì eÐnai peperasmèno sto ^peiro.

Qrhsimopoi¸ntac tic exis¸seic Euler-Lagrange gia thn dr^sh 2.1.22 paÐrnoume thn exÐswsh kÐnhshc

$$\frac{\partial^2 x}{\partial\omega^2} - \frac{\partial^2 x}{\partial u^2} + k^2e^{2u}x = 0 \tag{2.1.24}$$

Se mia maÔrh trÔpa, aut h exÐswsh eÐnai proseggistik kai isqÔei sthn perioq $2GM <$ r . $3GM$. Dokim^zoume lÔseic thc morf c

$$\varphi \propto x(u)e^{irt} = x(u)e^{ir4GM\omega} = x(u)e^{i\lambda\omega} \tag{2.1.25}$$

kai paÐrnoume

$$\lambda^2 x = -\frac{\partial^2 x}{\partial u^2} + k^2e^{2u}x \tag{2.1.26}$$

H stroform l sqetÐzetai me ton kumat^rijmo k mèsw thc sqèshc

$$\vec{L} = \vec{r}\times\vec{p} \Rightarrow l = |k|R_s = 2GM|k| \tag{2.1.27}$$

ParathroÔme ìti to dunamikì èqei thn morf $V \propto l^2$ ìpwc kai sthn perÐptwsh thc gewmetrÐac Schwarzschild. Epomènwc gia th perioq kont^ ston orÐzonta katal goume sto sumpèrasma ìti kb^nta enèrgeiac me meg^lh stroform den mporoÔn na diafÔgoun kai pèftoun mèsa sthn maÔrh trÔpa.

Aut^ ta apotelèsmata ja qrhsimeÔsoun argìtera ìtan melet soume thn aktinobolÐa Hawking, h opoÐa apoteleÐtai apì swmatÐdia pou par^gontai polÔ kont^ ston orÐzonta kai diafeÔgoun sto ^peiro.

2.2 Sun^rthsh epimerismoÔ kbantik¸n pedÐwn

OrÐzoume thn sun^rthsh epimerismoÔ Z

$$z = \sum_i e^{-\frac{E_i}{T}} \tag{2.2.1}$$

ìpou ajroÐzoume wc proc ìlec tic katast^seic tou sust matoc. H pijanìthta to sÔsthma na èqei enèrgeia E dÐnetai apì thn èkfrash

$$P_E = \frac{e^{-\frac{E}{T}}}{z} \tag{2.2.2}$$

ìpou T eÐnai h jermokrasÐa tou sust matoc.

'Estw ìti èqoume èna bajmwtì pedÐo Klein-Gordon (mpozonikì pedÐo). Mia genik idiokat^stash thc enèrgeiac eÐnai h akìloujh

$$\left(\prod_i (\alpha^{\dagger}_{\vec{p}_i})^{n_i}\right) |0\rangle \tag{2.2.3}$$

H kat^stash èqei enèrgeia $E = \sum_i n_i \omega_{p_i}$ kai orm $\vec{P} = \sum_i n_i \vec{p_i}$ ìpou n_i eÐnai to pl joc twn swmatidÐwn me enèrgeia E_i kai orm $\vec{p_i}$. H sun^rthsh epimerismoÔ tou pedÐou paÐrnei thn morf

$$z = \prod_i \frac{1}{1 - e^{-\frac{\omega_i}{T}}} \tag{2.2.4}$$

ìpou o deÐkthc i paÐrnei ìlec tic dunatèc timèc thc orm c.

H eleÔjerh enèrgeia F prosdiorÐzetai apì thn sqèsh

$$z = e^{-\frac{F}{T}}$$

kai isoÔtai me

$$F = -TV \int \frac{d^3\vec{p}}{(2\pi)^3} \log\left(\frac{1}{1 - e^{-\frac{\omega_{\vec{p}}}{T}}}\right) \tag{2.2.5}$$

ìpou V o ìgkoc tou q¸rou. Gia $m = 0$ h enèrgeia isoÔtai me to mètro thc orm c kai pa➀rnoume

$$F = -\frac{VT^4}{60\pi^2} \tag{2.2.6}$$

H mèsh enèrgeia thc katanom c prosdior➀zetai apì thn sqèsh

$$\langle E\rangle = U = \frac{\partial}{\partial\beta}(\beta F) \tag{2.2.7}$$

ìpou $\beta = 1/T$ kai h puknìthta enèrgeiac isoÔtai me

$$\rho = \frac{\langle E\rangle}{V} = \frac{\pi^2}{30}T^4 \tag{2.2.8}$$

H p➀esh d➀netai apì thn èkfrash

$$P = \frac{\partial F}{\partial V} = \frac{\rho}{3} \tag{2.2.9}$$

H ➀dia sqèsh sundèei thn p➀esh kai thn puknìthta enèrgeiac sto aèrio fwton➀wn. Ax➀zei na shmeiwje➀ ìti h puknìthta enèrgeiac e➀nai h mis apì ìti h ant➀stoiqh sto aèrio fwton➀wn kai o lìgoc e➀nai ìti ta fwtìnia èqoun 2 anex^rthtec sunist¸sec pìlwshc.

2.2.1 Sun^rthsh epimerismoÔ me sunarthsiak^ oloklhr¸mata

Jèloume na upolog➀soume thn sun^rthsh epimerismoÔ enìc sust matoc se jermokras➀a $T = 1/\beta$

$$z(\beta) = \sum_{E_i} e^{-\beta E_i} = \mathrm{Tr}\left(e^{-\widehat{H}\beta}\right) \tag{2.2.10}$$

Qrhsimopoi¸ntac wc b^sh tic idiokatast^seic tou telest $\widehat{H}$ kai ergazìmenoi sthn eikìna Schrodinger (ìpou oi telestèc e➀nai qronoanex^rthtoi kai oi katast^seic qronoexarthmènec) br➀skoume ìti

$$z(\beta) = \int dq \langle q| e^{-\widehat{H}\beta} |q\rangle = \int [dq]_p e^{-S_E} \tag{2.2.11}$$

ìpou o de➀kthc p dhl¸nei ^jroish wc proc periodik^ monop^tia. H per➀odoc e➀nai ➀sh me β $\left(q(t_E + \beta) = q(t_E)\right)$, ètsi ¸ste na anapar^getai h sqèsh 2.2.10. Sumpera➀noume ìti to sunarthsiakì olokl rwma upologÐzetai se pollaplìthta thc morf c $S^1 \times R^3$, ìpou o EukleÐdeioc qrìnoc sumpagopoieÐtai se kÔklo me perÐodo to antÐstrofo thc jermokrasÐac.

Ta apotelèsmata genikeÔontai kai sthn perÐptwsh kbantik¸n pedÐwn se epaf me dexamen jermìthtac [2]. Gia to pedÐo Klein-Gordon èqoume

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\varphi}{\partial t_E}\right)^2 - \frac{1}{2}\left(\nabla\varphi\right)^2 - \frac{1}{2}m^2\varphi^2 = -\mathcal{L}_E \tag{2.2.12}$$

H EukleÐdeia drˆsh isoÔtai me

$$S_E = \int dt_E d^3\vec{x}\left(\mathcal{L}_E\right) = -iS \tag{2.2.13}$$

EpÐshc apaitoÔme ta pedÐa na eÐnai periodikˆ wc proc ton EukleÐdeio qrìno.

BrÐskoume to plˆtoc metˆbashc enìc pedÐou apì mia arqik katˆstash me $t = t_i$ se mia telik katˆstash me $t = t_f$ me èna sunarthsiakì olokl rwma, ajroÐzontac wc proc ìlec tic diatˆxeic pedÐou pou ikanopoioÔn thn arqik kai telik sunj kh:

$$\int [d\varphi]_{\varphi_i(\vec{x})}^{\varphi_f(\vec{x})} e^{iS}, \quad S = \int_{t_i}^{t_f} dt \int d^3\vec{x}\,\mathcal{L}\,[\varphi, \partial_\mu\varphi] \tag{2.2.14}$$

2.2.2 Jemeli¸dhc katˆstash sust matoc

Mia katˆstash mporeÐ na grafteÐ sthn morf

$$\left|\bar{\Psi}\right\rangle = \sum_E |E\rangle\left\langle E|\bar{\Psi}\right\rangle \tag{2.2.15}$$

PaÐrnontac to ìrio mikr c jermokrasÐac $\beta \to \infty$ thc akìloujhc posìthtac èqoume

$$\lim_{\beta\to\infty}\sum_E e^{-\beta E}|E\rangle\left\langle E|\bar{\Psi}\right\rangle = N|0\rangle \tag{2.2.16}$$

ìpou N eÐnai o parˆgontac kanonikopoÐhshc kai $|0\rangle$ h basik katˆstash tou sust matoc. Autì sumbaÐnei giatÐ sto ìrio $\beta \to \infty$ epibi¸nei mìno o ìroc me thn elˆqisth enèrgeia. Qrhsimopoi¸ntac thn sqèsh $\Psi[\varphi(\vec{x})] = \langle\varphi(\vec{x})|0\rangle$ gia thn kumatosunˆrthsh thc basik c katˆstashc tou pedÐou, paÐrnoume

$$N\Psi[\varphi_0(\vec{x})] = \lim_{\beta\to\infty}\langle\varphi_0|\,e^{-\beta H}\left|\bar{\Psi}\right\rangle = \lim_{\beta\to\infty}\langle\varphi_0|\,e^{-\beta H}|\varphi_i\rangle \tag{2.2.17}$$

kai ètsi

$$\Psi[\varphi_0(\vec{x})] = \frac{1}{N}\lim_{\beta\to\infty}\langle\varphi_0|\,e^{-\beta H}|\varphi_i\rangle = \int[d\varphi(t<0)]\,e^{-S_E} \tag{2.2.18}$$

ìpou $\varphi(\vec{x}, t = 0) = \varphi_0$. H kumatosun^rthsh pou perigr^fei thn jemeli¸dh kat^stash mporeÐ na perigrafeÐ apì èna sunarthsiakì olokl rwma ston misì eukleÐdeio q¸ro (je¸rhma Feynman-Hellman).

2.3 JermokrasÐa pedÐwn se kampulwmènouc qwrìqronouc

2.3.1 Qwrìqronoc Rindler

Arqik^ ja melet soume ton qwrìqrono Rindler. QrhsimopoioÔme tic uperbolikèc suntetagmènec ρ, ω pou orÐzontai wc

$$t = \rho \sin \omega, \quad x = \rho \cos \omega \tag{2.3.1}$$

To apeirostì di^sthma isoÔtai me

$$dS^2 = -\rho^2 d\omega^2 + d\rho^2 \tag{2.3.2}$$

SuneqÐzoume analutik^ ston eukleÐdeio qrìno, $\varphi = -i\omega$, kai paÐrnoume

$$dS^2 = \rho^2 d\varphi^2 + d\rho^2 \tag{2.3.3}$$

O EukleÐdeioc qrìnoc φ prèpei na eÐnai periodikìc, me perÐodo 2π, ¸ste na apofeÔgetai mia kwnik anwmalÐa. ParathroÔme ìti parìlo pou o qwrìqronoc Rindler apoteleÐ mìno èna tm ma tou qwrìqronou Minkowski, o EukleÐdeioc q¸roc eÐnai pl rhc kai perilamb^nei olìklhro to EukleÐdeio epÐpedo.

To olikì pedÐo φ_0 an^getai se dÔo anex^rthta pedÐa pou orÐzontai stic perioq $x > 0$ (φ_R) kai $x < 0$ (φ_L) antÐstoiqa

$$\varphi_0 = (\varphi_R, \varphi_L) \tag{2.3.4}$$

Tìte h kumatosun^rthsh thc basik c kat^stashc $\Psi[\varphi_0]$ dÐnetai apì thn èkfrash

$$\Psi[\varphi_R, \varphi_L] = N \int [d\varphi(t_E < 0)]\, e^{-S_E} = N \langle \varphi_R | e^{-\pi H_\omega} | \varphi_L \rangle \tag{2.3.5}$$

me $\varphi(t_E = 0) = (\varphi_R, \varphi_L)$. Kaj¸c o EukleÐdeioc qrìnoc metab^lletai apì 0 mèqri π, to pedÐo φ_L exelÐssetai sto pedÐo φ_R. To sunarthsiakì olokl rwma ston misì EukleÐdeio q¸ro mporeÐ na idwjeÐ wc to pl^toc pijanìthtac gia th met^bash aut .

O pÐnakac puknìthtac pou perigrˆfei thn perioq sthn opoÐa èqei prìsbash o parathrht c Rindler $(x > 0)$ dÐnetai apì thn èkfrash

$$\rho_R(\varphi_R, \varphi_R^{'}) = \int [d\varphi_L] \, \Psi\,[\varphi_R, \varphi_L] \, \Psi^* \left[\varphi_R^{'}, \varphi_L\right] \tag{2.3.6}$$

Qrhsimopoi¸ntac thn sqèsh 2.3.5, brÐskoume

$$\rho_R = N^2 e^{-2\pi H_\omega} \tag{2.3.7}$$

Autìc eÐnai ènac jermikìc pÐnakac puknìthtac me jermokrasÐa

$$T\omega = \frac{1}{2\pi} \tag{2.3.8}$$

'Estw o parathrht c Rindler akoloujeÐ thn kosmik gramm $\rho = \bar{\rho}$. H kanonik jermokrasÐa pou katagrˆfetai sto jermìmetro tou isoÔtai me

$$T_R = \frac{1}{2\pi\bar{\rho}}$$

H jermokrasÐa eÐnai anˆlogh thc epitˆqunshc tou parathrht . O parathrht c Rindler antilambˆnetai ton q¸ro gemˆto me èna jermì aèrio swmatidÐwn. Ta eikonikˆ swmatÐdia pou parˆgontai lìgw kbantik¸n diakumˆnsewn tou kenoÔ den mporoÔn na metrhjoÔn apì ton parathrht Minkowski exaitÐac thc arq c thc abebaiìthtac enèrgeiac-qrìnou. Sthn perÐptwsh tou parathrht Rindler, ta swmatÐdia pou parˆgontai ston pareljontikì orÐzonta, lìgw tou fainomènou thc diastol c tou qrìnou, zoun gia megˆlo qronikì diˆsthma, kai ètsi mporeÐ na ta antilhfjeÐ.

2.3.2 Qwrìqronoc Schwarzschild

Qrhsimopoi¸ntac ton EukleÐdeio qrìno sth gewmetrÐa Schwarzschild, $t_E = -it$, paÐrnoume:

$$dS^2 = \left(1 - \frac{2GM}{r}\right) dt_E^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2.3.9}$$

H exwterik perioq $r \geq 2GM$ kalÔptei olìklhro ton EukleÐdeio q¸ro.

Gia gia na apofÔgoume thn ìpoia kwnik anwmalÐa gia $r \to 2GM$ prèpei na epibˆloume periodikìthta ston EukleÐdeio qrìno kai na epilèxoume katˆllhla thn perÐodo. Gia ton skopì autì arkeÐ na exetˆsoume thn perÐptwsh tou qwrìqrono Rindler, afoÔ kontˆ ston orÐzonta o qwrìqronoc proseggÐzetai apì ton qwrìqrono Rindler. Gi' autì jètoume

$$\frac{t_E}{4GM} = \varphi \tag{2.3.10}$$

kai afoÔ h suntetagmènh φ prèpei na èqei perÐodo 2π, brÐskoume thn perÐodo tou EukleÐdeiou qrìnou t_E

$$\beta = 8\pi GM \tag{2.3.11}$$

ìpou β to antÐstrofo thc jermokrasÐac (se mon^dec ìpou h stajer^ Boltzmann isoÔtai me th mon^da).

Sthn asumptwtik perioq $(r \to \infty)$, paÐrnoume

$$dS^2 = dt_E^2 + dr^2 = 16G^2M^2d\varphi^2 + dr^2 \tag{2.3.12}$$

pou eÐnai metrik epif^neiac kulÐndrou me aktÐna $4GM$ (perÐmetro $8\pi GM$). ExaitÐac thc periodikìthtac tou EukleÐdeiou qrìnou ìla ta sunarthsiak^ oloklhr¸mata ston q¸ro autì ekfr^zoun jermikèc sunart seic susqètishc.

Kaj¸c $r \to 2GM$, $\beta \to 0$, kai epomènwc h jermokrasÐa apeirÐzetai ston orÐzonta. Sthn asumptwtik perioq paÐrnoume $T = 1/8\pi GM$ pou eÐnai h jermokrasÐa Hawking [6, 7].

Gia na katano soume thn proèleush thc jermokrasÐac prèpei na melet soume tic kbantikèc diakum^nseic tou kenoÔ. Oi anamenìmenec timèc twn diafìrwn pedÐwn sto kenì mhdenÐzontai, $\left(\langle\varphi(x)\rangle = 0\right)$, all^ up^rqoun ^llec sunart seic susqètishc pou sundèontai me tic kbantikèc diakum^nseic, oi opoÐec den mhdenÐzontai. Oi diakum^nseic tou kenoÔ den mporoÔn na metrhjoÔn eujèwc, ìmwc ephre^zoun ^llec parathr simec posìthtec. Se mia kbantik diakÔmansh pou antistoiqeÐ sth dhmiourgÐa zeÔgouc swmatidÐou-antiswmatidÐou, ta swmatÐdia prèpei na epanasundejoÔn kai na katastrafoÔn, afoÔ h kat^stash tou kenoÔ eÐnai mia eustaj c kat^stash. 'Otan to zeÔgoc dhmiourghjeÐ kont^ ston orÐzonta, up^rqei pijanìthta to swmatÐdio me arnhtik enèrgeia na pèsei mèsa sthn maÔrh trÔpa kai to swmatÐdio me jetik enèrgeia na diafeÔgei sto ^peiro. SwmatÐdia me arnhtik enèrgeia mporoÔn na up^rxoun wc pragmatik^ swmatÐdia sto eswterikì thc maÔrhc trÔpac epeid to barutikì dunamikì sthn anwmalÐa eÐnai polÔ arnhtikì kajist¸ntac thn sunolik arnhtik enèrgeia epitrept . Ta swmatÐdia pou diafeÔgoun sto ^peiro (se apousÐa dunamikoÔ) prèpei na eÐnai swmatÐdia me jetik enèrgeia kai apoteloÔn thn jermik aktinobolÐa Hawking.

To fainìmeno autì sumbaÐnei mìno stic maÔrec trÔpec, afoÔ sthn epif^neia astèrwn (me aktÐna megalÔterh thc barutik c touc aktÐnac) h enèrgeia hremÐac twn swmatidÐwn eÐnai megalÔterh kat'apìluth tim apì th dunamik touc enèrgeia. Epomènwc h sunolik enèrgeia den mporeÐ na eÐnai arnhtik .

Kont¸ ston orÐzonta h jermokrasÐa dÐnetai apì thn èkfrash thc jermokrasÐac Rindler

$$T_{eff} = \frac{T_H}{\sqrt{1-\frac{2GM}{\bar{r}}}}, \quad \bar{\rho} = 2\sqrt{2GM(\bar{r}-2GM)} \tag{2.3.13}$$

ìpou T_H h jermokrasÐa Hawking.

2.4 EntropÐa sÔmplexhc

OrÐzoume thn entropÐa sÔmplexhc (entropÐa Bonnemann) wc [6]

$$S = -\operatorname{tr}[\rho_A \log \rho_A] = -\operatorname{tr}[\rho_B \log \rho_B] \tag{2.4.1}$$

Oi mh mhdenikèc idiotimèc twn pin¸kwn puknìthtac ρ_A kai ρ_B isoÔntai. Se mia b¸sh ìpou o ρ_A eÐnai diag¸nioc, h entropÐa sÔmplexhc dÐnetai apì thn èkfrash

$$S = -\sum_i (\rho_i \log \rho_i) \tag{2.4.2}$$

ìpou ρ_i eÐnai oi idiotimèc tou pÐnaka ρ_A. ParathroÔme ìti oi mhdenikèc idiotimèc èqoun mhdenik suneisfor¸ kaj¸c sto ìrio $\rho_i \to 0$, h posìthta $\rho_i \log \rho_i$ teÐnei sto 0.

Sthn perÐptwsh miac kajar c kat¸stashc up¸rqei mia mh mhdenik idiotim Ðsh me 1 kai oi upìloipèc eÐnai mhdèn. 'Etsi h entropÐa sÔmplexhc mhdenÐzetai. H entropÐa sÔmplexhc gÐnetai mègisth ìtan ìlec oi idiotimèc eÐnai Ðsec. Tìte èqoume pl rh ¨gnoia gia to sÔsthma. Se aut thn perÐptwsh, ìlec oi idiotimèc paÐrnoun thn tim $\rho_i = 1/D_A$, ìpou D_A h diastatikìthta tou sust matoc A. O pÐnakac puknìthtac paÐrnei thn morf

$$\rho_A = \frac{1}{D_A} \mathbb{1}_{D_A}$$

ìpou $\mathbb{1}_{D_A}$ o monadiaÐoc pÐnakac. H entropÐa sÔmplexhc gÐnetai Ðsh me

$$S = S_{max} = -D_A \frac{1}{D_A} \log \frac{1}{D_A} = \log D_A \tag{2.4.3}$$

H entropÐa sÔmplexhc den eÐnai 'prosjetik '. Sthn perÐptwsh susthm¸twn pou brÐskontai se sÔmplexh, h entropÐa sÔmplexhc tou kajenìc xeqwrist¸ den eÐnai 0, all¸ tou sunolikoÔ sust matoc mhdenÐzetai epeid to sunolikì sÔsthma brÐsketai se mia kajar kat¸stash.

OrÐzoume wc mètro thc plhroforÐac I thn posìthta

$$I = S_{max} - S \tag{2.4.4}$$

ParathroÔme ìti gia mia kajar kat¸stash èqoume mègisth plhroforÐa, en¸ h plhroforÐa mhdenÐzetai gia mia jermik kat¸stash gia thn opoÐa èqoume pl rh ¨gnoia, dhlad $S =$

S_{max}.
An èqoume èna jermikì p□naka puknìthtac, tìte

$$\rho = \frac{e^{-\beta \widehat{H}}}{\mathrm{Tr}\, e^{-\beta \widehat{H}}} = \frac{1}{z} \sum_E e^{-\beta E} \left|E\right\rangle \left\langle E\right| \tag{2.4.5}$$

kai h entrop□a sÔmplexhc d□netai apì thn èkfrash

$$S = -\,\mathrm{tr}\,\rho \log \rho = \log z + \frac{1}{T} \sum_E E e^{-\beta E} \frac{1}{z} = \frac{-F}{T} - \frac{1}{T} U \tag{2.4.6}$$

Aut e□nai h entrop□a thc kanonik c sullog c (U e□nai h mèsh enèrgeia kai F h eleÔjerh enèrgeia).

2.5 Jermik entrop□a

'Estw ìti èqoume èna sÔsthma me polloÔc bajmoÔc eleujer□ac pou èqei meg^lh enèrgeia. Tìte mporoÔme na upojèsoume ìti h enèrgeia moir^zetai isìposa stouc bajmoÔc eleujer□ac me b^sh thn arq thc isokatanom c thc enèrgeiac. Diamer□zoume to sÔsthma S se n mikrìtera uposust mata Σ_i, me polÔ ligìterouc bajmoÔc eleujer□ac apì autoÔc tou sunolikoÔ sust matoc. Tìte kat^ mèso ìro, to k^je uposÔsthma èqei enèrgeia $\epsilon = E_\Sigma/n$. To uposÔsthma perigr^fetai me polÔ kal prosèggish apì èna jermikì p□naka puknìthtac

$$\rho_i = \frac{e^{-\beta H_i}}{z_i} \tag{2.5.1}$$

Gia èna tètoio uposÔsthma h jermik entrop□a isoÔtai me thn entrop□a sÔmplexhc $S^i_{Th} = S_i$.

H sunolik jermik entrop□a tou sust matoc d□netai apì thn èkfrash

$$S_{Th} = \sum_i S_i \tag{2.5.2}$$

H jermik entrop□a e□nai 'prosjetik '.

H plhrofor□a pou perièqetai sto sÔsthma e□nai

$$I = S_{Th} - S \tag{2.5.3}$$

Gia to olikì sÔsthma èqoume mègisth plhrofor□a afoÔ h entrop□a sÔmplexhc e□nai mhdèn $(S = 0)$, en¸ gia èna uposÔsthma Σ_i h plhrofor□a e□nai mhdèn afoÔ $S^i_{Th} = S_i$.

T¸ra èstw ìti èqoume èna uposÔsthma Σ_1, me arqikì mègejoc autì tou Σ_i, to opo□o me suneq trìpo megal¸nei mèqri na kalÔyei olìklhro to sÔsthma S. Ac doÔme pwc metab^lletai h plhrofor□a pou perièqei to sÔsthma Σ_1. H plhrofor□a akolouje□ thn kampÔlh Page [8, 9], sÔmfwna me thn opo—a h plhrofor—a tou sust matoc Σ_1 paramènei mhdenik mèqri na apokt sei touc misoÔc per—pou bajmoÔc eleujer—ac tou sust matoc S. Tìte apokt˜ èna bit plhrofor—ac. Epomènwc gia $\Sigma_1 < \Sigma/2$, $S(\Sigma_1) = S_{Th}(\Sigma_1)$, kai gia $\Sigma_1 > \Sigma/2$ èqoume

$$S(\Sigma_1) = S(\Sigma - \Sigma_1) = S_{Th}(\Sigma - \Sigma_1) \tag{2.5.4}$$

afoÔ oi entrop—ec sÔmplexhc twn 2 uposusthm˜twn isoÔntai, kai e˜n $\Sigma_1 > \Sigma/2$, tìte $\Sigma - \Sigma_1 < \Sigma/2$. H plhrofor—a pou perièqei to sÔsthma Σ_1 e—nai

$$\begin{aligned} I &= S_{Th}(\Sigma_1) - S(\Sigma_1) = S_{Th}(\Sigma_1) - S_{Th}(\Sigma - \Sigma_1) \\ &= fS_{Th}(\Sigma) - (f-1)S_{Th}(\Sigma) = (2f-1)I_\Sigma \end{aligned} \tag{2.5.5}$$

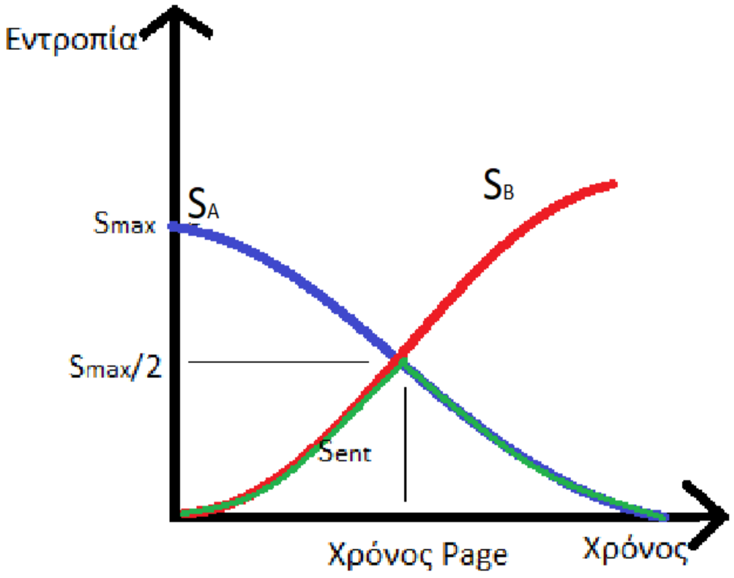
ìpou f e—nai to phl—ko tou arijmoÔ twn bajm¸n eleujer—ac tou sust matoc Σ_1 wc proc ton arijmì twn bajm¸n eleujer—ac tou sust matoc S. O tÔpoc isqÔei gia $f > 1/2$.

2.5.1 Par˜deigma

Ac exet˜soume èna par˜deigma gia to pwc exel—ssontai h entrop—a sÔmplexhc kai h jermik entrop—a. Arqik˜ èqoume èna sÔsthma pou br—sketai se kajar kat˜stash kai apotele—tai apì to sÔsthma A, to perib˜llon, kai to sÔsthma B, èna kout— me mia bìmba. Oi bajmo— eleujer—ac antistoiqoÔn se talantwtèc tou fwtonikoÔ ped—ou. Se k˜poia qronik stigm h bìmba ekr gnutai diege—rontac touc talantwtèc sto eswterikì tou koutioÔ. ProtoÔ arq—soun na diafeÔgoun fwtìnia apì to kout—, h entrop—a sÔmplexhc metaxÔ twn susthm˜twn A kai B e—nai mhdèn, en¸ h jermik entrop—a tou sust matoc B e—nai meg˜lh lìgw thc jermìthtac pou apèkthse. 'Otan arq—soun na diafeÔgoun fwtìnia sto perib˜llon, h entrop—a sÔmplexhc arq—zei na aux˜netai. Pa—rnei th mègisth tim thc ìtan h jermik entrop—a twn A kai B exiswjoÔn, kai mhden—zetai kai p˜li ìtan ìla ta fwtìnia diafÔgoun sto perib˜llon. Sto tèloc h jermik entrop—a tou sust matoc B mhden—zetai afoÔ h jermìthta tou èqei diafÔgei sto perib˜llon kai ìlh h jermik entrop—a br—sketai ston sÔsthma R. (H sunolik jermik entrop—a den mpore— na meiwje— se ant—jesh me thn entrop—a sÔmplexhc). H qronik exèlixh twn di˜forwn entropi¸n tou sust matoc apeikon—zetai sthn eikìna 2.5.1.

2.6 Entrop—a sÔmplexhc sto qwrìqrono Rindler kai sth maÔrh trÔpa

H basik kat˜stash ston qwrìqrono Minkowski den taut—zetai me thn basik kat˜stash tou qwrìqronou Rindler, afoÔ o parathrht c Rindler antilamb˜netai to q¸ro gem˜to



Eikona 2.5.1: Oi diakekommènec grammèc antistoiqoÔn sthn jermik entropÐa tou sust -matoc R kai B, en¸ h maÔrh suneqìmenh kampÔlh eÐnai h kampÔlh Page, h kampÔlh thc entropÐac sÔmplexhc. O qrìnoc Page eÐnai o qrìnoc pou pernˆ gia na arqÐsei h plhro-forÐa na metafèretai apì to sÔsthma B sto sÔsthma R [10].

me jermikì aèrio swmatidÐwn, se antÐjesh me ton parathrht Minkowski o opoÐoc antil-ambˆnetai ton q¸ro kenì.

H entropÐa gia ˆmazo pedÐo se mh mhdenik jermokrasÐa dÐnetai apì thn sqèsh [2]

$$S = \frac{2\pi^2}{45} V T^3 \tag{2.6.1}$$

ìpou V o ìgkoc kai T h jermokrasÐa. Gia thn perioq kontˆ ston orÐzonta, h jermokrasÐa eÐnai $T = 1/2\pi\rho$. 'Etsi h entropÐa se stoiqei¸dh ìgko ΔV kontˆ ston orÐzonta eÐnai

$$\Delta S = \frac{2\pi^2}{45} A \Delta\rho \left(\frac{1}{2\pi\rho}\right)^3 \tag{2.6.2}$$

h opoÐa apeirÐzetai gia $\rho \to 0$. Epibˆlloume ìrio apokop c gia tic mikrèc timèc tou ρ kai brÐskoume

$$S = \frac{1}{180\pi} \int_{\epsilon}^{\infty} \frac{\Delta\rho}{\rho^3} = \frac{1}{90\pi\epsilon^2} \tag{2.6.3}$$

Gia timèc tou ε thc tˆxhc tou m kouc Planck, h entropÐa sÔmplexhc eÐnai thc tˆxhc thc en-tropÐac Bekenstein-Hawking pou eÐnai h sunolik entropÐa thc maÔrhc trÔpac. (Aut prèpei na eÐnai megalÔterh apì thn entropÐa sÔmplexhc). Se aut thn apìstash, h jermokrasÐa eÐnai thc tˆxhc thc jermokrasÐac Planck, gi' autì anamènoume h kbantik jewrÐa pedÐou na mhn perigrˆfei swstˆ to sÔsthma.

2.7 Enèrgeia Rindler maÔrhc trÔpac

Jèloume na broÔme thn enèrgeia tou orÐzonta miac maÔrhc trÔpac sto sÔsthma tou parathrht Rindler. Gia na to k^noume autì prèpei na qrhsimopoi soume tic sqèseic metaxÔ twn telest¸n pou sundèontai me tic di^forec metr simec posìthtec. O telest c met^jeshc metaxÔ thc qamiltonian c enìc sust matoc kai tou qrìnou eÐnai

$$[H, t] = i \tag{2.7.1}$$

ìpou H kai t h qamhltonian kai o qrìnoc Schwarzschild antÐstoiqa. Efarmìzontac aut thn sqèsh gia thn qamiltonian kai ton qrìno Rindler paÐrnoume

$$[H_R, \omega] = i \tag{2.7.2}$$

Qrhsimopoi¸ntac thn sqèsh $\omega = t/4MG$ – h enèrgeia thc maÔrhc trÔpac M apoteleÐ idiotim thc qamiltonian c sth gewmetrÐa Schwarzschild – brÐskoume

$$[H_R, t] = i4GM = i4GH \tag{2.7.3}$$

Qrhsimopoi¸ntac thn sqèsh

$$[A^n, B] = [A, B]\, nA^{n-1} \equiv [A, B]\, \frac{\partial A^n}{\partial A}, \quad \big[A^k, [A, B]\big] = 0 \tag{2.7.4}$$

brÐskoume

$$\frac{\partial H_R}{\partial H} = 4GM \tag{2.7.5}$$

Epomènwc

$$H_R = 2GH^2 \Rightarrow E_R = 2GM^2 \tag{2.7.6}$$

ìpou E_R eÐnai h enèrgeia thc maÔrhc trÔpac sto sÔsthma tou parathrht Ridler.

Qrhsimopoi¸ntac thn èkfrash gia thn jermokrasÐa Rindler $T = 1/2\pi$ kai thn èkfrash thc entropÐac $S = 4\pi GM^2$ sunart sei thc m^zac, blèpoume ìti ikanopoieÐtai o pr¸toc jermodunamikìc nìmoc sto sÔsthma Rindler: $dE_R = TdS$. Epomènwc o orÐzontac èqei jermikèc idiìthtec eÐte melethjeÐ apì th skopi^ tou parathrht Schwarzschild eÐte apì th skopi^ tou parathrht Rindler.

2.8 Qrìnoc zw c maÔrhc trÔpac

Gia na doÔme ìti ìntoc mia maÔrh trÔpa exaÔl¸netai. Ja ektim soume ton rujmì exaÔlwshc mia maÔrhc trÔpac. Ja jewr soume ìti h m^za eÐnai polÔ meg^lh (astronomik¸n diast^sewn), ¸ste na proseggÐsoume thn perioq kont^ ston orÐzonta me thn metrik Rindler. EpÐshc ja jewr soume ìti ta swmatÐdia pou diafeÔgoun kai prokaloÔn thn ap¸leia enèrgeiac eÐnai s-swmatÐdia (me mhdenik stroform), epeid swmatÐdia me mh mhdenik stroform prèpei na uperphd soun megalÔtero fr^gma dunamikoÔ gia na diafÔgoun sto ^peiro, kai h suneisfor^ touc eÐnai mikr .

AfoÔ to fr^gma gia ta s-swmatÐdia eÐnai thc t^xhc thc jermokrasÐac, tìte diafeÔgei èna kb^nto an^ mon^da qrìnou $\sim 1/GM$ (se qrìno Schwarzschild). K^je kb^nto èqei enèrgeia thc t^xhc thc jermokrasÐac Hawking $1/8\pi GM$. Epomènwc h enèrgeia an^ mon^da qrìnou pou diafeÔgei sto ^peiro eÐnai Ðsh me

$$L = \frac{c}{M^2G^2} = -\frac{dM}{dt} \tag{2.8.1}$$

ìpou c mia stajer^ (kat^ prosèggish stajer^ gia meg^lec m^zec). BrÐskoume loipìn ìti o qrìnoc exaÖlwshc eÐnai

$$t_{\varepsilon\xi} \sim M^3G^2 \tag{2.8.2}$$

H maÔrhc trÔpa aktinoboleÐ ìpwc èna mèlan s¸ma

$$L \sim T^4A \tag{2.8.3}$$

H aktinobolÐa thc maÔrhc trÔpac exart^tai apì thn jermokrasÐa me ton Ðdio trìpo ìpwc h aktinobolÐa tou liou. O mhqanismìc ìmwc pou prokaleÐ thn aktinobolÐa Hawking eÐnai polÔ diaforetikìc. To m koc kÔmatoc thc aktinobolÐac tou liou eÐnai polÔ mikrìtero apì thn aktÐna tou liou, epomènwc mporoÔme na exag^goume plhroforÐec sqetik^ me ton lio. To antÐstoiqo m koc kÔmatoc gia thn aktinobolÐa miac maÔrhc trÔpac eÐnai thc t^xhc thc aktÐnac thc, epomènwc den mporoÔme na diakrÐnoume leptomèreiec gia th dom thc. Autì sumfwneÐ me to ìti h aktinobolÐa Hawking eÐnai kajar^ jermik kai den metafèrei plhroforÐa.

2.9 H lektrikèc idiìthtec orÐzonta maÔrhc trÔpac

'Opwc eÐdame prohgoumènwc an qrhsimopoi soume thn kbantik jewrÐa pedÐou gia na upologÐsoume thn entropÐa sÔmplexhc sunant^me apeirismoÔc pou proèrqontai apì thn perioq kont^ ston orÐzonta. 'Enac trìpoc na antimetwpÐsoume autì to prìblhma eÐnai na antikatast soume ton majhmatikì orÐzonta $r = 2GM$ me mia membr^nh se apìstash perÐpou l_p apì ton majhmatikì orÐzonta. Me autì ton trìpo apofeÔgoume touc apeirismoÔc. Profan¸c h prosèggis aut den dÐnei akrib apotelèsmata, all^ mac bohj^ na katano soume pollèc apì tic idiìthtec tou orÐzonta.

Se mia fortismènh maÔrh trÔpa h metrik kontˆ ston orÐzonta eÐnai

$$d\tau^2 = \rho^2 d\omega^2 - d\rho^2 - dx_\perp^2 \tag{2.9.1}$$

H drˆsh tou hlektromagnhtikoÔ pedÐou eÐnai

$$S = \int \frac{\sqrt{-g}}{16\pi} \left[g^{\mu\nu} g^{\sigma\tau} F_{\mu\sigma} F_{\nu\tau} + j^\mu A_\mu \right] d\omega d\rho d^2 x_\perp \tag{2.9.2}$$

Qrhsimopoi¸ntac tic sqèseic

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \dot{\vec{A}} = \frac{\partial \vec{A}}{\partial \omega}, \quad \varphi = -A_0, \quad \partial_\mu j^\mu = 0 \tag{2.9.3}$$

$$\vec{E} = -\boldsymbol{\nabla}\varphi - \dot{\vec{A}}, \quad \vec{B} = \boldsymbol{\nabla} \times \vec{A} \tag{2.9.4}$$

brÐskoume

$$S = \int \left[\frac{1}{8\pi} \left(\frac{\vec{A} + \boldsymbol{\nabla}\varphi^2}{\rho} - \rho \left(\boldsymbol{\nabla} \times \vec{A} \right)^2 \right) + \vec{j} \cdot \vec{A} \right] d\omega d\rho dx_\perp^2 \tag{2.9.5}$$

Oi exis¸seic tou Maxwell eÐnai

$$\begin{aligned}
\frac{1}{\rho}\dot{\vec{E}} - \vec{\nabla} \times (\rho \vec{B}) &= -4\pi \vec{j} \\
\dot{\vec{B}} + \vec{\nabla} \times \vec{E} &= 0 \\
\vec{\nabla}. \left(\frac{1}{\rho} \vec{E} \right) &= 4\pi j_0 \\
\vec{\nabla}.\vec{B} &= 0
\end{aligned} \tag{2.9.6}$$

Ja exetˆsoume arqikˆ hlektrostatikˆ pedÐa sto sÔsthma tou epitaqunìmenou parathrht Rindler. EpÐshc jewroÔme ìti ta pedÐa sto eswterikì tou orÐzonta mhdenÐzontai. H epifaneiak puknìthta fortÐou orÐzetai wc

$$\sigma = \frac{1}{4\pi\rho} E_\rho|_{\rho=\rho_0} = \frac{1}{4\pi\rho} \partial_\rho \Phi|_{\rho=\rho_0} \tag{2.9.7}$$

Sthn bajmÐda Coulomb ($\vec{\nabla} \cdot \vec{A} = 0$) paÐrnoume

$$\partial_\rho^2 \Phi - \frac{1}{\rho} \partial_\rho \Phi = -\nabla_\perp^2 \Phi \tag{2.9.8}$$

Sto ìrio $\rho \to 0$, to dunamikì paÐrnei thn morf

$$\Phi \sim \rho^a, \quad a = 0, 2 \Rightarrow \Phi = F(x_\perp) + \rho^2 G(x_\perp), \quad \nabla_\perp^2 F = 0 \tag{2.9.9}$$

jewr¸ntac ìti o orÐzontac eÐnai sumpag c kai amel¸ntac ìrouc an^logouc tou ρ^2 (afoÔ ston orÐzonta h apìstash r eÐnai polÔ mikr), brÐskoume ìti ston orÐzonta to dunamikì eÐnai stajerì. O orÐzontac sumperifèretai san ènac agwgìc.

Gia èna hlektromagnhtikì kÔma pou diadÐdetai aktinik^ proc ton orÐzonta paÐrnoume

$$\dot{B}_x = \partial_\rho E_y, \quad \dot{B}_y = -\partial_\rho E_x, \quad \frac{1}{\rho}\dot{E}_x = -\partial_\rho(\rho B_y), \quad \frac{1}{\rho}\dot{E}_y = \partial_\rho(\rho B_x) \tag{2.9.10}$$

kai ètsi

$$j_x = \frac{1}{4\pi}E_x, \; j_y = \frac{1}{4\pi}E_y \tag{2.9.11}$$

Epomènwc katal goume sto sumpèrasma ìti o orÐzontac sumperifèretai ìpwc ènac wmikìc agwgìc me agwgimìthta $1/4\pi$.

'Estw ìti b^zoume èna fortismèno swmatÐdio na kineÐtai aktinik^ proc ston orÐzonta. Jamelet soume thn asumptwtik sumperifor^ kaj¸c to swmatÐdio plhsi^zei ton orÐzonta. Sthn perioq kont^ ston orÐzonta h metrik proseggÐzetai me thn metrik Rindler. H aktinik kateÔjunsh ρ sumpÐptei me ton ^xona twn z. Me b^sh to hlektrikì pedÐo shmeiakoÔ fortÐou pou brÐsketai se apìstash z_0 apì ag¸gimo epÐpedo ^peirou embadoÔ, paÐrnoume

$$E_\rho = E_z = \frac{e(z-z_0)}{\left[(z-z_0)^2 + x_\perp^2\right]^{\frac{3}{2}}} = \frac{(\rho\cosh\omega - z_0)}{\left[(\rho\cosh\omega - z_0)^2 + x_\perp^2\right]^{\frac{3}{2}}} \tag{2.9.12}$$

H puknìthta fortÐou prosdiorÐzetai apì thn sqèsh $\sigma = E_\rho/4\pi\rho|_{\rho=\rho_0}$. Gia meg^la ω, $\cosh\omega \simeq e^\omega$. OrÐzontac thn metablht $\Psi = e^\omega x$ paÐrnoume

$$\sigma = \frac{e}{4\pi}\frac{e^{-2\omega}}{(\epsilon_0^2 + \Psi_\perp^2)} \tag{2.9.13}$$

O parathrht c Rindler antilamb^netai ton orÐzonta wc mia jermik kai ag¸gimh membr^nh. AxioshmeÐwto eÐnai ìti tic idiìthtec autèc pou prosdÐdei ston orÐzonta o exwterikìc parathrht c Rindler (jermokrasÐa, entropÐa, hlektrik agwgimìthta k.t.l.), den tic antilamb^netai ènac parathrht c pou pèftei eleÔjera.

Kef^laio 3

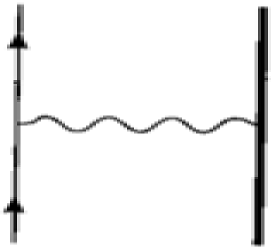
AktinobolÐa, upèrujra jewr mata kai diathroÔmena reÔmata sthn QED

3.1 Diorj¸seic aktinobolÐac sthn QED

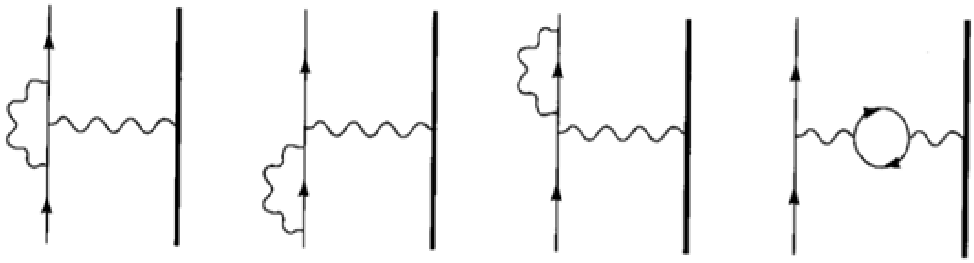
'Otan upologÐzoume to pl^toc mia diergasÐac sthn QED èqoume thn kÔria suneisfor^ pou proèrqetai apì ta diagr^mmata qamhlìterhc t^xhc (diagr^mmata dèntrou sthn perÐptwsh pou èqoume dÔo arqik^ swmatÐdia) kai tic diorj¸seic aktinobolÐac (radiative corrections) pou upologÐzontai apì diagr^mmata uyhlìterhc t^xhc. Aut^ ta diagr^mmata pollèc forèc perièqoun brìqouc kai ìpwc ja doÔme qrei^zontai eidik metaqeÐrish kaj¸c apeirÐzontai. Oi diorj¸seic aktinobolÐac sthn QED perilamb^noun diagr^mmata ekpomp c fwtonÐwn (aktinobolÐa bremsstrahlung). Se autì to kef^laio ja anafèroume perilhptik^ tic dÔo phgèc diorj¸sewn sta diagr^mmata Feynman gia thn QED, kai ja doÔme pwc sundu^zontai gia na anairèsoun touc di^forouc apeirismoÔc [11].

Ja meletsoume èna aplì par^deigma, thn skèdash enìc hlektronÐou apì èna barÔ pur na. O barÔc pur nac den suneisfèrei sthn ekpomp aktinobolÐac, kai ètsi mìno to hlektrìnio ekpèmyei fwtìnia. Sthn klasik perÐptwsh, h epit^qunsh tou fortismènou pur na kat^ thn skèdash eÐnai mikr exaitÐac thc sqetik^ meg^lhc m^zac tou pur na. Sunep¸c aktinoboleÐ el^qista, afoÔ h aktinobolÐa eÐnai an^logh thc epit^qunshc. Apì thn ^llh, h epit^qunsh tou hlektronÐou eÐnai meg^lh, kai ètsi h aktinobolÐa tou den eÐnai amelhtèa.

To di^gramma dèntrou thc skèdashc hlektronÐou apì barÔ pur na apeikonÐzetai sthn eikìna 3.1.1, en¸ sthn eikìna 3.1.2 apeikonÐzontai ta diagr^mmata deÔterhc t^xhc. To pr¸to di^gramma thc eikìnac 3.1.2 eÐnai gnwstì wc diìrjwsh koruf c kai dÐnei thn kÔria suneisfor^ sthn diìrjwsh deÔterhc t^xhc. Ta dÔo epìmena diagr^mmata eÐnai diagr^mmata



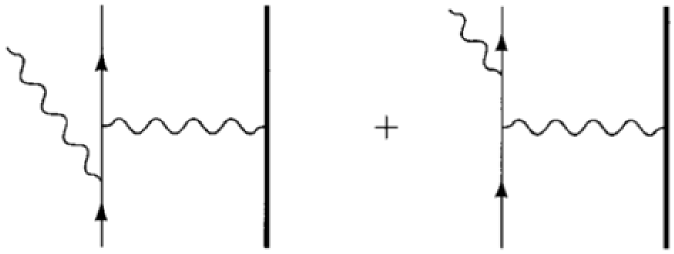
Eikona 3.1.1: Di^gramma dèntrou gia thn skèdash hlektronÐou apì barÔ pur na [11]. H èntonh gramm antiproswpeÔei ton barÔ pur na pou antall^ssei èna fwtìnio me to hlektrìnio.



Eikona 3.1.2: Diagr^mmata deÔterhc t^xhc gia thn skèdash hlektronÐou apì barÔ pur na [11]. Up^rqoun tèssera diagr^mmata pou suneisfèroun kai èqoun ìla eswterikoÔc brìqouc.

diìrjwshc exwterik c gramm c (perièqoun touc legìmenouc asÔndetouc brìqouc) kai oi suneisforèc touc mporoÔn na afairejoÔn me th diadikasÐa thc epakanonikopoÐhshc. To teleutaÐo eÐnai to di^gramma pìlwshc. K^je èna apì ta poio p^nw diagr^mmata perièqei èna olokl rwma wc proc thn aprosdiìristh orm brìqou to opoÐo apoklÐnei sto uperi¸dec ìrio $k \to \infty$. 'Omwc oi parathr simec posìthtec ìpwc h energìc diatom thc skèdashc prèpei na eÐnai peperasmènec. Eutuq¸c oi apeirismoÐ twn oloklhrwm^twn mporoÔn na afairejoÔn me th diadikasÐa thc epakanonikopoÐhshc (h QED eÐnai epanakanonikopoi shmh jewrÐa).

To pr¸to di^gramma parousi^zei epÐshc upèrujrec apoklÐseic, dhlad to olokl rwma wc proc thn orm brìqou apoklÐnei kai sto ìrio $k \to 0$. AutoÐ oi apeirismoÐ anairoÔntai lamb^nontac upìyin ta diagr^mmata ekpomp c fwtonÐwn (diagr^mmata Bremsstrahlung) thc eikìnac 3.1.3. Ta diagr^mmata aut^ apoklÐnoun sto ìrio ìpou h enèrgeia tou ekpempìmenou fwtonÐou mhdenÐzetai. 'Omwc se autì to ìrio ta ekpempìmena fwtìnia èqoun tìso qamhl enèrgeia pou den mporoÔn na aniqneutoÔn apì touc aniqneutèc sta peir^mata. Ta pio p^nw diagr^mmata suneisfèroun sthn parathroÔmenh energì diatom thc skèdashc hlektronÐou me barÔ pur na. Oi apeirismoÐ pou parousi^zoun aut^ ta diagr^mmata anairoÔn touc upèrujrouc apeirismoÔc twn diagramm^twn thc eikìnac 3.1.2 [11].



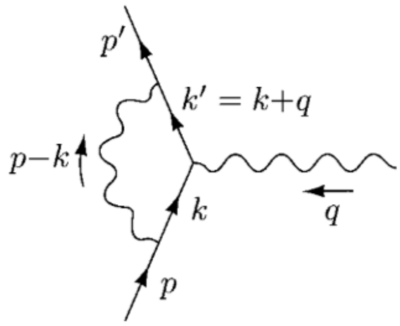
Eikona 3.1.3: Diagr^mmata Bremsstrahlung gia thn skèdash hlektronÐou apì barÔ pur na [11]. ParathroÔme ìti kai ta dÔo diagr^mmata èqoun mia exerqìmenh gramm fwtonÐou pou den up^rqei sto di^gramma dèntrou kÔriac t^xhc.

3.1.1 UpologismoÐ se brìqouc

A foÔ eÐdame poiotik^ pwc qeirizìmaste diagr^mmata uyhlìterhc t^xhc, ja anafèroume se suntomÐa pwc mporoÔn na upologistoÔn autèc oi diorj¸seic. Gia na k^noume upologismoÔc se brìqouc prèpei na akolouj soume ta akìlouja b mata:

1) Sqediasmìc twn diagramm^twn kai eÔresh tou pl^touc (dec parart mata).
2) Eisagwg paramètrwn Feynman, ¸ste na sunduastoÔn oi paronomastèc twn diadot¸n.
3) Sumpl rwsh twn tetrag¸nwn twn paronomast¸n metatopÐzontac kat^llhla thn orm brìqou se mia nèa orm ℓ (ìpwc sto pio k^tw par^deigma).
4) Allag morf c arijmht¸n ¸ste na perilamb^nei dun^meic thc orm c ℓ. AgnooÔme perittèc dun^meic tou ℓ kai gr^foume tic ^rtiec dun^meic tou ℓ aporrof¸ntac touc deÐktec stic posìthtec $g^{\mu\nu}$.
5) Metatrop oloklhrwm^twn orm c se tetradi^stata EukleÐdeia oloklhr¸mata me qr sh strof c Wick.

AkoloÔjwc efarmìzoume ta pio p^nw sto di^gramma diìrjwshc koruf c. Arqik^ orÐ-



Eikona 3.1.4: Diagr^mmata diìrjwshc koruf c [11].

zoume thn sun^rthsh $-ie\Gamma^\mu(p',p)$ wc to ^jroisma thc koruf c qamhlìterhc t^xhc kai thc diìrjwshc: $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$. Efarmìzontac touc kanìnec Feynman gia to pio p^nw par^deigma brÐskoume se t^xh α (stajer^ lept c uf c)

$$
\begin{aligned}
&\bar{u}(p')\delta\Gamma^{\mu}(p',p)u(p) \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')(-ie\gamma^{\nu})\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon}\gamma^{\mu}\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^{\rho})u(p) \\
&= 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\not{k}\gamma^{\mu}\not{k}'+m^2\gamma^{\mu}-2m(k+k')^{\mu}\right]u(p)}{\left((k-p)^2+i\epsilon\right)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}
\end{aligned}
\tag{3.1.1}
$$

O ìroc $i\epsilon$ ston paronomast den mporeÐ na agnohjeÐ kaj¸c eÐnai aparaÐthtoc gia na upologisteÐ to olokl rwma thc orm c brìqou.

Efarmìzontac thn mèjodo twn paramètrwn Feynman ston paronomast paÐrnoume

$$
\frac{1}{\left((k-p)^2+i\epsilon\right)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)} = \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3} \tag{3.1.2}
$$

ìpou o paronomast c D dÐnetai apì thn èkfrash

$$
\begin{aligned}
D &= x(k^2-m^2)+y(k'^2-m^2)+z(k-p)^2+(x+y+z)i\epsilon \\
&= k^2+2k.(yq-zp)+yq^2+zp^2-(x+y)m^2+i\epsilon
\end{aligned}
\tag{3.1.3}
$$

ìpou $k'=k+q$. Sumplhr¸noume ta tetr^gwna ston paronomast jètontac

$$
\ell \equiv k+yq-zp \tag{3.1.4}
$$

kai ètsi brÐskoume

$$
D = \ell^2-\Delta+i\epsilon \tag{3.1.5}
$$

ìpou

$$
\Delta \equiv -xyq^2+(1-z)^2m^2 \tag{3.1.6}
$$

Epeid $q^2<0$ gia skèdash, Δ jetik posìthta, kai ètsi mporeÐ na idwjeÐ wc energìc m^za.

Sth sunèqeia ekfr^zoume ton arijmht sunart sei thc orm c ℓ. Epeid h posìthta D exart^tai mìno apì to mètro thc orm c ℓ, isqÔoun oi sqèseic

$$
\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^{\mu}}{D^3}=0 \tag{3.1.7}
$$

$$
\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^{\mu}\ell^{\nu}}{D^3}=\int\frac{d^4\ell}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}\ell^2}{D^3} \tag{3.1.8}
$$

Oi perittèc dun^meic thc orm c ℓ den suneisfèroun lìgw summetrÐac. H deÔterh tautìthta apodeiknÔetai qrhsimopoi¸ntac to gegonìc ìti aut mhdenÐzetai ektìc kai an $\mu=\nu$. H sunj kh sunalloiìthtac k^tw apì metasqhmatismoÔc Lorentz epib^llei o arijmht c na eÐnai an^logoc me ton antÐstrofo thc metrik c $g^{\mu\nu}$. Gia na prosdiorÐsoume ton suntelest analogÐac paÐrnoume to Ðqnoc me th metrik $g_{\mu\nu}$.

Qrhsimopoi¸ntac tic tautìthtec autèc gr^foume ton arijmht sthn morf

$$\begin{aligned}
&\bar{u}(p')\big[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\big]u(p) \\
= \bar{u}(p')&\left[-\frac{1}{2}\gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu\big((1-y)\not{q} + z\not{p}\big) + m^2\gamma^\mu - 2m\big((1-2y)q^\mu + 2zp^\mu\big)\right]u(p)
\end{aligned} \tag{3.1.9}$$

Qrhsimopoi¸ntac thn ^lgebra Dirac, paÐrnoume thn èkfrash

$$\begin{aligned}
&\bar{u}(p')\big[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\big]u(p) \\
&= \bar{u}(p')\Big[\gamma^\mu\big(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\big) \\
&\quad + (p^\mu + p'^\mu)mz(z-1) + q^\mu m(z-2)(x-y)\Big]u(p)
\end{aligned} \tag{3.1.10}$$

O suntelest c thc orm c q^μ mhdenÐzetai lìgw summetrÐac. Qrhsimopoi¸ntac thn tautìthta Gordon

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{3.1.11}$$

katal goume sto apotèlesma

$$\begin{aligned}
\bar{u}(p')\delta\Gamma^\mu(p',p)u(p) &= \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3} \\
\times\bar{u}(p')\Big[\gamma^\mu\big(-\frac{1}{2}\ell^2 &+ (1-x)(1-y)q^2 + (1-4z-z^2)m^2\big) \\
&+ \frac{i\sigma^{\mu\nu}q_\nu}{2}m^2 z(1-z)\Big]u(p)
\end{aligned} \tag{3.1.12}$$

Gia na oloklhr¸soume ton upologismì prèpei na ektelèsoume to olokl rwma wc proc ℓ. O eukolìteroc trìpoc eÐnai na qrhsimopoi soume strof Wick ston EukleÐdeio q¸ro. OrÐzoume $i\ell_E \equiv \ell^0$, $\vec{\ell}_E = \vec{\ell}$ ìpou ℓ_E h orm sto EukleÐdeio q¸ro. To olokl rwma gÐnetai

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(2\pi)^4}\int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^3}{(\ell_E^2+\Delta)^m} \tag{3.1.13}$$

Sto par^deigma pou meletoÔme $m = 3$. To olokl rwma wc proc thn epif^neia sfaÐrac ston tetradi^stato q¸ro dÐnei èna par^gonta $2\pi^2$. To olokl rwma paÐrnei thn morf

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^m} = \frac{i(-1)^m}{(2\pi)^4}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}} \tag{3.1.14}$$

Epðshc sthn perðptwsh $m > 3$, mporoÔme na qrhsimopoi soume thn akìloujh èkfrash

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^m} = \frac{i(-1)^{m-1}}{(2\pi)^4}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}} \tag{3.1.15}$$

Gia $m = 3$ to olokl rwma apoklðnei.

Gia na to apofÔgoume autì ja eis^goume mia par^metro apokop c pou ja k^nei to olokl rwma peperasmèno. Jètoume loipìn

$$\frac{1}{(k-p)^2+i\epsilon} \to \frac{1}{(k-p)^2+i\epsilon} - \frac{1}{(k-p)^2-\Lambda^2+i\epsilon} \tag{3.1.16}$$

ìpou to Λ eðnai mia polÔ meg^lh m^za. To olokl rwma den ephre^zetai gia mikr^ k. MporoÔme na skeftoÔme ton deÔtero ìro wc ton diadìth enìc barioÔ fwtonðou, tou opoðou afairoÔme thn suneisfor^. Ta apotelèsmata den all^zoun gia ton ìro autì me thn diafor^ ìti

$$\Delta \to \Delta_\Lambda = \Delta + z\Lambda^2 \tag{3.1.17}$$

To olokl rwma paðrnei thn morf

$$\begin{aligned}&\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{(\ell^2-\Delta)^m} - \frac{\ell 2}{(\ell^2-\Delta_\Lambda)^m}\right)\\ &= \frac{i}{(2\pi)^4}\int_0^\infty d\ell_E^2\left(\frac{\ell_E^4}{(\ell_E^2+\Delta)^m} - \frac{\ell_E^4}{(\ell_E^2+\Delta_\Lambda)^m}\right) = \frac{i}{(2\pi)^4}\log\left(\frac{\Delta_\Lambda}{\Delta}\right)\end{aligned} \tag{3.1.18}$$

Aut h teqnik onom^zetai omalopoðhsh Pauli-Villars.

Qrhsimopoi¸ntac ta oloklhr¸mata 3.1.14 kai 3.1.18, h èkfrash 3.1.12 an^getai sthn akìloujh

$$\begin{aligned}&\frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\\ &\times\bar{u}(p')\Bigg(\gamma^\mu\left[\log\frac{z\Lambda^2}{\Delta} + \frac{1}{\Delta}\Big((1-x)(1-y)q^2 + (1-4z+z^2)m^2\Big)\right]\\ &+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\frac{1}{\Delta}2m^2z(1-z)\right]\Bigg)u(p)\end{aligned} \tag{3.1.19}$$

Qrhsimopoi¸ntac autì to apotèlesma mporoÔme na upologðsoume thn diìrjwsh ston par^gonta g tou hlektronðou [11]

$$a_e = \frac{g-2}{2} \text{ w } 0.0011614 \tag{3.1.20}$$

Autì to apotèlesma upologðsthke pr¸th for^ apì ton Schwinger to 1948. H peiramatik tim eðnai 0.0011597. Lamb^nontac upìyh diorj¸seic an¸terhc t^xhc, prokÔptei akìma megalÔterh sumfwnða metaxÔ thc peiramatik c kai thc jewrhtik c tim c.

3.2 Upèrujra jewr mata sthn QED kai sth barÔthta

Se autì to kef^laio ja melet soume pwc qeirizìmaste tic upèrujrec apoklðseic pou prokÔptoun apì upèrujra (soft) fwtìnia kai barutìnia sthn QED kai sth barÔthta [12].

3.2.1 Eisagwg enìc upèrujrou fwtonðou barutonðou

Arqik^ ja exet^soume thn perðptwsh ìpou prosjètoume se èna di^gramma Feynman èna exwterikì upèrujro swmatðdio. E^n prosjèsoume èna fwtìnio me orm q se mia gramm exerqìmenou fortismènou swmatidðou, ja prèpei na pollaplasi^soume to sunolikì pl^toc tou diagr^mmatoc me èna epiprìsjeto diadìth me orm $p+q$ kai mia epiprìsjeth koruf gia thn met^bash $p \to p+q$. E^n h exwterik gramm tou fwtonðou en¸netai me èna eiserqìmeno fortismèno swmatðdio tìte o epiprìsjetoc diadìthc èqei orm $p-q$ kai h koruf perigr^fei th met^bash $p \to p-q$.

Gia par^deigma, e^n to fortismèno swmatðdio èqei mhdenikì spin, tìte o epiprìsjetoc par^gontac eðnai

$$i(2\pi)^4 e(2p^\mu + \eta q^\mu)\frac{-i}{(2\pi)^4}\frac{1}{(p+\eta q)^2 - m^2 - i\epsilon} \tag{3.2.1}$$

ìpou $\eta = +1$ gia exerqìmeno fortismèno swmatðdio kai $\eta = -1$ gia eiserqìmeno. Sto ìrio $q \to 0$ h exðswsh 3.2.1 gðnetai

$$\frac{e\eta p^\mu}{p.q - i\eta\epsilon} \tag{3.2.2}$$

ìpou qrhsimopoi same thn sqèsh $p^2 - m^2 = 0$.

H sqèsh 3.2.2 genikeÔetai kai gia fortismèna swmatðdia me mh mhdenikì spin [12]. 'Etsi gia èna aujaðreto di^gramma, èna epiplèon exwterikì upèrujro fwtìnio suneisfèrei me ton par^gonta

$$\sum_n \frac{e_n \eta_n p_n^\mu}{p_n.q - i\eta_n\epsilon} \tag{3.2.3}$$

ìpou to ^jroisma perilamb^nei ìlec tic exwterikèc grammèc tou diagr^mmatoc.

Gia thn perÐptwsh enìc barutonÐou paÐrnoume

$$\frac{1}{2}i(2\pi)^4(8\pi G)^{1/2}(2p^\mu+\eta q^\nu)(2p^\nu+\eta q^\nu)\frac{-i}{(2\pi)^4}\frac{1}{(p+\eta q)^2-m^2-i\epsilon} \tag{3.2.4}$$

ìpou m,n eÐnai oi deÐktec pìlwshc tou barutonÐou. Sto ìrio $q\to 0$ èqoume

$$\frac{(8\pi G)^{1/2}\eta p^\mu p^\nu}{p.q-i\eta\epsilon} \tag{3.2.5}$$

To apotèlesma autì isqÔei anex^rthta apì to spin [12]. H koruf pou perigr^fei thn allhlepÐdrash tou barutonÐou me to exwterikì swmatÐdio eÐnai an^logh tou par^gonta $(8\pi G)^{1/2}$. SumperaÐnoume loipìn ìti gia èna tuqaÐo di^gramma èqoume

$$(8\pi G)^{1/2}\sum_n\frac{e_n\eta_n p_n^\mu p_n^\nu}{p_n.q-i\eta_n\epsilon} \tag{3.2.6}$$

ìpou to ^jroisma eÐnai wc proc ìlec tic exwterikèc grammèc.

3.2.2 N upèrujra swmatÐdia

MporoÔme na genikeÔsoume ta pio p^nw apotelèsmata sthn perÐptwsh poll¸n upèrujrwn swmatidÐwn (fwtìnia barutìnia). Sthn perÐptwsh aut , prèpei na prosjèsoume èna par^gonta thc morf c 3.2.2 3.2.5 gia k^je epiprìsjeto upèrujro swmatÐdio. E^n to swmatÐdio r sundèetai teleutaÐo sthn exwterik gramm fortismènou swmatidÐou, to s proteleutaÐo k.t.l, ja èqoume mia suneisfor^ apì touc diadìtec:

$$\frac{1}{pq_r-i\eta\epsilon}\frac{1}{p(q_r+q_s)-i\eta\epsilon}\ldots \tag{3.2.7}$$

ìpou o k^je diadìthc emplèkei diaforetik orm lìgw diat rhshc thc orm c. Gia N sunolik^ swmatÐdia, o pio p^nw ìroc prèpei na ajroisteÐ wc proc $N!$ sunduasmoÔc. To apotèlesma tou ajroÐsmatoc eÐnai

$$\frac{1}{pq_1-i\eta\epsilon}\frac{1}{pq_2-i\eta\epsilon}\ldots \tag{3.2.8}$$

Parìmoio apotèlesma paÐrnoume sthn perÐptwsh poll¸n barutonÐwn [12].

3.2.3 ApoklÐseic lìgw eikonik¸n upèrujrwn swmatidÐwn

Oi upèrujrec apoklÐseic qwrÐzontai se dÔo kathgorÐec, tic eikonikèc pou sqetÐzontai me swmatÐdia eswterik¸n brìqwn kai tic pragmatikèc pou sqetÐzontai me exwterik^ swmatÐdia. Arqik^ ja exet^soume tic eikonikèc apoklÐseic.

OrÐzoume wc upèrujro eikonikì swmatÐdio to fwtìnio barutìnio to opoÐo en¸netai me dÔo exwterikèc grammèc fortismènwn swmatidÐwn kai èqei orm mikrìterh apì Λ, $q \leq \Lambda$. EpÐshc orÐzoume èna upèrujro ìrio apokop c λ, ètsi ¸ste na omalopoi soume touc upèrujrouc apeirismoÔc: $q \geq \lambda$. Sundèontac N eikonik^ fwtìnia paÐrnoume N zeÔgh apì par^gontec thc morf c 3.2.3, me to k^je zeÔgoc na sundèetai me èna diadìth fwtonÐou

$$\frac{-i}{(2\pi)^4}\frac{g_{\mu\nu}}{q^2-i\epsilon} \tag{3.2.9}$$

ìpou ajroÐzoume wc proc touc deÐktec pìlwshc kai oloklhr¸noume wc proc tic ormèc q. Prèpei epÐshc na diairèsoume me èna par^gonta $N!$ pou isoÔtai me ton par^gonta summetrÐac tou diagr^mmatoc. To telikì apotèlesma pou paÐrnoume eÐnai

$$\frac{1}{N!}\left[\int_{\lambda}^{\Lambda} d^4qA(q)\right]^N \tag{3.2.10}$$

ìpou

$$A(q) = \frac{-1}{(2\pi)^4(q^2-i\epsilon)}\sum_{nm}\frac{e_ne_m\eta_n\eta_m(p_n{\cdot}p_m)}{(p_n{\cdot}q-i\eta_n\epsilon)(-p_m{\cdot}q-i\eta_m\epsilon)} \tag{3.2.11}$$

To prìshmo plhn ston deÔtero par^gonta tou paronomast sto ^jroisma ofeÐletai sto ìti oi diadìtec twn eikonik¸n fwtonÐwn sundèontai se dÔo exwterikèc grammèc, kai ètsi e^n h orm pou eisèrqetai sto èna ^kro eÐnai jetik (ekpomp fwtonÐou), sto ^llo ^kro eÐnai arnhtik (aporrìfhsh fwtonÐou).

To pl^toc skèdashc gia mia tuqaÐa diergasÐa mporeÐ na ekfrasteÐ sthn morf

$$S_{\beta\alpha} = S^0_{\beta\alpha}e^{\frac{1}{2}\int_{\lambda}^{\Lambda}d^4qA(q)} \tag{3.2.12}$$

ìpou $S^0_{\beta\alpha}$ to pl^toc qwrÐc eikonik^ upèrujra fwtìnia. H pijanìthta an^ mon^da qrìnou gia th met^bashc $\alpha\to\beta$ eÐnai an^logh tou tetag¸nou thc apìluthc tim c tou pl^touc, $\sim \Gamma_{\beta\alpha} = |S_{\beta\alpha}|^2$:

$$\Gamma_{\beta\alpha} = \Gamma^0_{\beta\alpha}e^{\mathrm{Re}\int_{\lambda}^{\Lambda}\mathrm{d^4qA(q)}} \tag{3.2.13}$$

To pragmatikì mèroc tou oloklhr¸matoc isoÔtai me

$$\mathrm{Re}\int_{\lambda}^{\Lambda}\mathrm{d^4qA(q)} = -\frac{1}{2(2\pi)^3}\int_{\lambda}^{\Lambda}\mathrm{d^4q}\delta(\mathrm{q^2})\sum_{\mathrm{nm}}\frac{\mathrm{e_ne_m}\eta_{\mathrm{n}}\eta_{\mathrm{m}}(\mathrm{p_n{\cdot}p_m})}{(\mathrm{p_n{\cdot}q})(\mathrm{p_m{\cdot}q})} = -\mathrm{A}\log\left(\frac{\Lambda}{\lambda}\right) \tag{3.2.14}$$

ìpou A mia jetik adi˜stath staj er˜

$$A \equiv \int d^2\Omega A(\widehat{q}) = \int d^2\Omega \frac{1}{2(2\pi)^3} \sum_{nm} \frac{e_n e_m \eta_n \eta_m (p_n \cdot p_m)}{(E_n - \vec{p}_n \cdot \widehat{q})(E_n - \vec{p}_m \cdot \widehat{q})}$$
$$= -\frac{1}{8\pi} \sum_{nm} e_n e_m \eta_n \eta_m \frac{1}{\beta_{nm}} \log\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right) \tag{3.2.15}$$

kai β_{nm} to mètro thc sqetik c taqÔthtac metaxÔ twn swmatidÐwn n kai m:

$$\beta_{nm} \equiv \left(1 - \frac{m_n^2 m_m^2}{(p_n \cdot p_m)^2}\right)^{\frac{1}{2}} \tag{3.2.16}$$

Qrhsimopoi¸ntac tic exis¸seic 3.2.13 kai 3.2.14, brÐskoume

$$\Gamma_{\beta\alpha} = \Gamma^0_{\beta\alpha} \left(\frac{\lambda}{\Lambda}\right)^A \tag{3.2.17}$$

ParomoÐwc gia ta barutìnia paÐrnoume

$$S_{\beta\alpha} = S^0_{\beta\alpha} e^{\frac{1}{2}\int_\lambda^\Lambda d^4 q B(q)} \tag{3.2.18}$$

ìpou to $B(q)$ h genÐkeush tou par˜gonta $A(q)$ gia ta barutìnia. O diadìthc tou barutonÐou eÐnai

$$\frac{-i}{2(2\pi)^4} \frac{g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma}}{q^2 - i\epsilon} \tag{3.2.19}$$

Akolouj¸ntac an˜loga b mata me thn perÐptwsh twn fwtonÐwn brÐskoume [12]

$$\Gamma_{\beta\alpha} = \Gamma_{\beta\alpha} \left(\frac{\lambda}{\Lambda}\right)^B \tag{3.2.20}$$

ìpou

$$B = \frac{G}{2\pi} \sum_{nm} \eta_n \eta_m m_n m_m \frac{1+\beta_{nm}^2}{\beta_{nm}(1+\beta_{nm}^2)^{1/2}} \log\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right) \tag{3.2.21}$$

Epeid $B > 0$ h pijanìthta met˜bashc mhdenÐzetai sto ìrio $\lambda \to 0$, ìpwc sumbaÐnei kai sthn perÐptwsh thc QED. To prìblhma lÔnetai lamb˜nontac upìyin thn ekpomp pragmatik¸n upèrujrwn fwtonÐwn kai barutonÐwn.

3.2.4 ApoklÐseic lìgw ekpomp c pragmatik¸n upèrujrwn swmatidÐwn

To pl˜toc ekpomp c N upèrujrwn pragmatik¸n fwtonÐwn se mia diergasÐa $\alpha \to \beta$ prokÔptei e˜n pollaplasi˜soume to pl˜toc skèdashc thc diergasÐac $\alpha \to \beta$ (qwrÐc aktinobolÐa) me èna par˜gonta thc morf c 3.2.3 kai thn antÐstoiqh 'kummatosun˜rthsh'

$$\frac{1}{\left((2\pi)^3 2|\vec{q}|\right)^{1/2}}\epsilon^*_\mu(\vec{q},h) \tag{3.2.22}$$

ìpou ϵ_μ to di˜nusma pìlwshc, $h = \pm 1$ h elikìthta kai $\vec{q}$ h orm tou ekpempìmenou fwtonÐou. 'Etsi brÐskoume to pl˜toc met˜bashc me ekpomp N fwtonÐwn

$$S^{ph}_{\beta\alpha}(12\ldots N) = S_{\beta\alpha}\prod_{r=1}^{N}\frac{1}{\left((2\pi)^3 2|\vec{q}|\right)^{1/2}}\sum_n \frac{\eta_n e_n \big[p_n.\epsilon^*(\vec{q}_r,h_r)\big]}{p_n.q_r} \tag{3.2.23}$$

Sthn perÐptwsh thc barÔthtac h 'kummatosun˜rthsh' paÐrnei th morf

$$\frac{1}{\left((2\pi)^3 2|\vec{q}|\right)^{1/2}}\epsilon^*_\mu(\vec{q},\pm 1)\epsilon^*_\nu(\vec{q},\pm 1) \tag{3.2.24}$$

kai to pl˜toc [12]

$$S^{gr}_{\beta\alpha}(12\ldots N) = S_{\beta\alpha}\prod_{r=1}^{N}\frac{1}{\left((2\pi)^3 2|\vec{q}|\right)^{1/2}}(8\pi G)^{1/2}\sum_n \frac{\eta_n \big[p_n.\epsilon^*(\vec{q}_r,\frac{1}{2}h_r)\big]^2}{p_n.q_r} \tag{3.2.25}$$

H pijanìthta an˜ mon˜da qrìnou kai stic dÔo peript¸seic prokÔptei e˜n ajroÐsoume wc proc tic elikìthtec twn ekpempìmenwn fwtonÐwn/barutonÐwn:

$$\Gamma^{ph}_{\beta\alpha}(\vec{q}_1\ldots\vec{q}_N)d^3\vec{q}_1\ldots d^3\vec{q}_N = \frac{1}{N!}\Gamma_{\beta\alpha}\prod_{r=1}^{N}\mathcal{A}(\vec{q}_r)d^3\vec{q}_r \tag{3.2.26}$$

$$\Gamma^{gr}_{\beta\alpha}(\vec{q}_1\ldots\vec{q}_N)d^3\vec{q}_1\ldots d^3\vec{q}_N = \frac{1}{N!}\Gamma_{\beta\alpha}\prod_{r=1}^{N}\mathcal{B}(\vec{q}_r)d^3\vec{q}_r \tag{3.2.27}$$

ìpou

$$\mathcal{A}(\vec{q}) = \frac{A(\widehat{q})}{|\vec{q}|^3} \tag{3.2.28}$$

H sun˜rthsh $A(\hat{q})$ dÐnetai apì thn èkfrash 3.2.15. ParomoÐwc gia ta barutìnia paÐrnoume

$$\mathcal{B}(\vec{q}) = \frac{B(\widehat{q})}{|\vec{q}|^3} \tag{3.2.29}$$

Oloklhr¸nontac tic ekfr^seic 3.2.26 kai 3.2.27 wc proc tic stereèc gwnðec paðrnoume

$$\Gamma^{ph}_{\beta\alpha}(\omega_1\ldots\omega_N)d\omega_1\ldots d\omega_N = \frac{A^N}{N!}\Gamma_{\beta\alpha}\prod_{r=1}^{N}\frac{d\omega_r}{\omega_r} \tag{3.2.30}$$

ìpou ω_r h enèrgeia tou fwtonðou r. Paromoðwc gia barutìnia

$$\Gamma^{gr}_{\beta\alpha}(\omega_1\ldots\omega_N)d\omega_1\ldots d\omega_N = \frac{B^N}{N!}\Gamma_{\beta\alpha}\prod_{r=1}^{N}\frac{d\omega_r}{\omega_r} \tag{3.2.31}$$

Oi ekfr^seic autèc deðqnoun ìti oi pijanìthtec (an^ mon^da qrìnou) ekpomp c fwtonðwn barutonðwn parousi^zoun logarijmikèc upèrujrec apoklðseic.

3.2.5 Upèrujra swmatðdia sunolik c enèrgeiac mikrìterhc apì E

Ja upologðsoume th sunolik pijanìthta an^ mon^da qrìnou, h diergasða $\alpha\to\beta$ na sunodeÔetai apì aujaðreto arijmì upèrujrwn swmatidðwn sunolik c enèrgeiac mikrìterhc apì E. H pijanìthta aut dðdetai apì thn èkfrash

$$\Gamma_{\beta\alpha}(\leq E) = \frac{1}{\pi}\sum_{N=0}^{\infty}\prod_{r=1}^{N}\int_{\lambda}^{E}d\omega_r\int_{-\infty}^{\infty}d\sigma\frac{\sin(E\sigma)}{\sigma}e^{i\sigma\sum_r\omega_r}\Gamma_{\beta\alpha}(\omega_1\ldots\omega_N) \tag{3.2.32}$$

ìpou qrhsimopoi same mia oloklhrwtik èkfrash gia thn sun^rthsh b matoc θ [12]. Sto ìrio $\lambda\to 0$

$$\int_{\lambda}^{E}\frac{d\omega}{\omega}e^{i\omega\sigma}\to\log\left(\frac{E}{\lambda}\right)+\int_{\lambda}^{E}d\omega(e^{i\omega\sigma}-1)+\mathcal{O}(\lambda) \tag{3.2.33}$$

kai ètsi paðrnoume

$$\Gamma^{ph}_{\beta\alpha}(\leq E) = \left(\frac{E}{\lambda}\right)^{A}b(A)\Gamma_{\beta\alpha} \tag{3.2.34}$$

$$\Gamma^{gr}_{\beta\alpha}(\leq E) = \left(\frac{E}{\lambda}\right)^{B}b(B)\Gamma_{\beta\alpha} \tag{3.2.35}$$

ìpou

$$b(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}d\sigma\frac{\sin(E\sigma)}{\sigma}e^{x\int_0^\infty\frac{d\omega}{\omega}(e^{i\omega\sigma}-1)} = 1-\frac{\pi^2}{12}x^2+\mathcal{O}(x^3) \tag{3.2.36}$$

AfoÔ ta A kai B eðnai jetik^ oi exis¸seic 3.2.34 kai 3.2.35 parousi^zoun upèrujrec apoklðseic sto ìrio $\lambda\to 0$.

3.2.6 AnaÐresh apoklÐsewn

Gia na oloklhr¸soume ton upologismì sundu^zoume tic exis¸seic 3.2.17 kai 3.2.34, kaj¸c epÐshc tic 3.2.20 kai 3.2.35, brÐskontac

$$\Gamma^{ph}_{\beta\alpha}(\leq E) = \left(\frac{E}{\Lambda}\right)^{A} b(A)\Gamma_{\beta\alpha} \tag{3.2.37}$$

$$\Gamma^{gr}_{\beta\alpha}(\leq E) = \left(\frac{E}{\Lambda}\right)^{B} b(B)\Gamma_{\beta\alpha} \tag{3.2.38}$$

Blèpoume ìti oi upèruj rec apoklÐseic pou proèrqontai apì ta eikonik^ upèruj ra swmatÐdia se eswterikoÔc brìqouc me autèc pou proèrqontai apì pragmatik^ exerqìmena upèruj ra swmatÐdia anairoÔntai, kai ètsi to apotèlesma eÐnai peperasmèno. H ex^rthsh apì to ìrio apokop c λ apaleÐfetai. H enèrgeia Λ oriojeteÐ thn enèrgeia twn upèruj rwn eikonik¸n swmatidÐwn.

3.3 Meg^loi metasqhmatismoÐ bajmÐdac (LGT)

Se autì to kef^laio meletoÔme mia meg^lh kathgorÐa metasqhmatism¸n bajmÐdac pou den mhdenÐzontai sto ^peiro. Klassik^ autoÐ oi metasqhmatismoÐ odhgoÔn se èna ^peiro arijmì diathroumènwn fortÐwn, kai ja anafèroume epipt¸seic pou èqoun aut^ ta fortÐa sthn kbantik perÐptwsh.

H lagkrazian gia ton hlektromagnhtismì èqei thn morf

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu \tag{3.3.1}$$

ìpou o tanust c $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ eÐnai amet^blhtoc k^tw apì metasqhmatismoÔc bajmÐdac $A'_\mu = A_\mu - \partial_\mu \Lambda$. H exÐswsh kÐnhshc eÐnai

$$\partial_\nu F^{\nu\mu} = -J^\mu \tag{3.3.2}$$

Wc proc èna topikì metasqhmatismì bajmÐdac

$$\delta A_\mu = -\partial_\mu \epsilon(x) \tag{3.3.3}$$

h metabol thc lagkrazian c gÐnetai

$$\delta L = -J^\mu(x)\partial_\mu \epsilon(x) = \partial_\mu\big(\epsilon(x)j(x)\big) \tag{3.3.4}$$

ìpou sto teleutaÐo b ma qrhsimopoi same thn exÐswsh sunèqeiac $\partial_\mu j^\mu = 0$ (diat rhsh hlektrikoÔ fortÐou).

Me b^sh to je¸rhma Noether paÐrnoume to akìloujo diathroÔmeno reÔma

$$j^\mu_\epsilon = \frac{\partial L}{\partial(\partial_\mu A_\nu)}\delta A_\nu + \epsilon j^\mu = F^{\mu\nu}\partial_\nu\epsilon + J^\mu\epsilon \tag{3.3.5}$$

ìpou qrhsimopoi same thn exÐswsh

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

K^nontac qr sh thc exÐswshc kÐnhshc mporoÔme na gr^youme to diathroÔmeno reÔma sthn morf

$$j^\mu_\epsilon = \partial_\nu(F^{\mu\nu}\epsilon) \tag{3.3.6}$$

pou exasfalÐzei thn diat rhsh $\partial_\mu j^\mu_\epsilon = 0$. Up^rqei èna tèteio reÔma gia k^je bajmÐda ϵ.

To diathroÔmeno fortÐo dÐnetai apì thn èkfrash

$$Q_\epsilon = \int d^3\vec{x} j^0_\epsilon \tag{3.3.7}$$

ìpou to qwrikì olokl rwma orÐzetai se olìklhro ton q¸ro. Sthn perÐptwsh pou h sun^rthsh ϵ den mhdenÐzetai sto ^peiro, all^ ekdhl¸nei gwniak ex^rthsh, to olokl rwma eÐnai mh mhdenikì. AfoÔ mporoÔme na epilèxoume mia aujaÐreth tètoia sun^rthsh $\epsilon(x)$, prokÔptei ^peiroc arijmìc diathroÔmenwn fortÐwn.

Ta diathroÔmena fortÐa kajorÐzoun idiìthtec tou pÐnaka skèdashc S pou sundèei tic arqikèc kai tic telikèc katast^seic se di^forec diergasÐec thc QED. To pl^toc skèdashc miac arqik c kat^stashc $|\Psi\rangle_{in}$ na p^ei se mia telik kat^stash $|\Psi\rangle_{out}$ dÐnetai apì thn èkfrash

$$_{out}\langle\Psi| S |\Psi\rangle_{in} \tag{3.3.8}$$

Sthn QED ta diathroÔmena fortÐa an^gontai se telestèc oi opoÐoi metatÐjentai me thn Qamiltonian . Gr^fontac Q^+_ϵ gia ton telest tou fortÐou se qrìno $T \to +\infty$ kai Q^-_ϵ gia $T \to -\infty$, gr^foume ton nìmo diat rhshc sthn morf miac tautìthtac Ward

$$_{out}\langle\Psi| (Q^+_\epsilon S - S Q^-_\epsilon) |\Psi\rangle_{in} = 0 \tag{3.3.9}$$

ApodeiknÔetai ìti autèc oi tautìthtec Ward isodunamoÔn me ta upèrujra jewr mata [4, 13, 14, 15, 12] pou kajorÐzoun thn ekpomp upèrujrwn fwtonÐwn kai touc upèrujrouc apeirismoÔc sta diagr^mmata Feynman.

Gia na ikanopoioÔntai oi nìmoi diat rhshc, prèpei na up^rqei ^peiroc arijmìc upèrujrwn fwtonÐwn sthn telik kat^stash, ¸ste to pl^toc na eÐnai mh mhdenikì. Enallaktik^ ntÔnoume ta fortismèna swmatÐdia thc arqik c kai thc telik c kat^stashc me èna nèfoc upèrujrwn fwtonÐwn [16]. Pr^gmati, oi kal^ orismènec asumptwtikèc katast^seic twn fortismènwn swmatidÐwn lamb^noun upìyin to hlektrikì pedÐo pou dhmiourgoÔn.

'Ara loipìn to pl^toc skèdashc metaxÔ katast^sewn me peperasmèno arijmì upèrujrwn fwtonÐwn mhdenÐzetai exaitÐac thc ekjetopoÐhshc twn eikonik¸n upèrujrwn apeirism¸n. Enallaktik^, o mhdenismìc twn plat¸n aut¸n mporeÐ na katanohjeÐ wc sunèpeia summetrÐac twn ^peirwn nìmwn diat rhshc thc QED [12].

Sthn perÐptwsh thc barÔthtac oi topikoÐ metasqhmatismoÐ eÐnai oi genikoÐ metasqhmatismoÐ suntetagmènwn. To pedÐo sto qwrìqrono eÐnai h metrik . Sthn genik sqetikìthta mporoÔme p^nta na all^xoume sÔsthma mèsw genik¸n metasqhmatism¸n suntetagmènwn me tic metr simec posìthtec na paramènoun amet^blhtec epeid ikanopoioÔn tanustikèc exis¸seic [1]. 'Opwc kai sthn perÐptwsh thc QED, up^rqoun ^peiroi nìmoi diat rhshc pou sundèontai me topikoÔc meg^louc metasqhmatismoÔc metafor^c kai peristrof c (supertranslations and superrotations) pou den mhdenÐzontai sto ^peiro. Oi nìmoi autoÐ prèpei na ikanopoioÔntai se ìlec tic diadikasÐec barutik¸n sked^sewn. Oi nìmoi odhgoÔn sta upèrujra jewr mata sth barÔthta [4]. Wc sunèpeia to pl^toc skèdashc metaxÔ katast^sewn pou perièqoun èna peperasmèno arijmì apì upèrujra barutìnia eÐnai mhdenikìc.

Meg^loi metasqhmatismoÐ suntetagmènwn up^rqoun kai stic mh abelianèc jewrÐec bajmÐdoc ìpwc h QCD kai h jewrÐa pedÐou twn asjen¸n allhlepidr^sewn. Sthn perÐptwsh thc QCD ta diathroÔmena fortÐa epib^lloun periorismoÔc stic sked^seic metaxÔ twn kou^rkc kai twn gklouonÐwn.

[1] Οι τανυστικές εξισώσεις ισχύουν σε όλα τα συστήματα αναφοράς συντεταγμένων.

Kef^laio 4

Par^doxa pou sunant^me stic maÔrec trÔpec

Se autì to kef^laio ja melet soume merik^ apì ta par^doxa pou prokÔptoun stic maÔrec trÔpec. H melèth aut¸n twn paradìxwn eÐnai polÔ shmantik gia thn katanìhsh thc barÔthtac kai to p¸c sqetÐzetai me tic ^llec allhlepidr^seic. Sto teleutaÐo mèroc tou kefalaÐou ja d¸soume idiaÐterh èmfash sto par^doxo diat rhshc thc plhroforÐac stic maÔrec trÔpec kai tic di^forec lÔseic pou èqoun protajeÐ, kaj¸c se mia apì autèc basÐzetai to deÔtero mèroc thc ergasÐac aut c.

4.1 FusikoÐ Nìmoi

Se autì to kef^laio parousi^zontai di^forec basikèc arqèc thc Fusik c p^nw stic opoÐec basÐzontai ta par^doxa kai oi lÔseic touc. Autèc eÐnai: 1)H arq thc diat rhshc thc kbantik c plhroforÐac, 2)h arq thc isodunamÐac kai 3)h arq thc mh klwnopoÐhshc thc kbantik c plhroforÐac. Ja anafèroume epÐshc thn arq thc sumplhrwmatikìthtac kai thn arq thc olografÐac pou sundèontai me thn perigraf susthm^twn kbantik c barÔthtac.

Me b^sh ta peiramatik^ dedomèna, h fÔsh perigr^fetai apì touc nìmouc thc kbantomhqanik c kai thc eidik c sqetikìthtac. Oi nìmoi thc kbantomhqanik c odhgoÔn sthn kbantik jewrÐa pedÐou pou perigr^fei ta fainìmena se enèrgeiec arket^ mikrìterec thc enèrgeiac Planck. 'Omwc mia kbantik jewrÐa pedÐou den mporeÐ na perigr^yei èna barutikì sÔsthma, kaj¸c h mia barutik jewrÐa pedÐou par^gei sto kbantikì epÐpedo apeirismoÔc, pou den mporoÔn na afairejoÔn me th diadikasÐa thc epanakanonikopoÐhshc.

4.1.1 Diat rhsh thc kbantik c plhroforÐac

Me bˆsh thn arq thc diat rhshc thc plhroforÐac, h plhroforÐa se èna apomonwmèno sÔsthma diathreÐtai dhlad den mporeÐ na qajeÐ oÔte na dhmiourghjeÐ.

'Estw loipìn ìti èqoume èna sÔsthma se mia arqik katˆstash. H exeligmènh katˆstash ikanopoieÐ thn exÐswsh Schrodinger

$$i\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle \tag{4.1.1}$$

ìpou H h qamiltonian tou sust matoc. Eˆn h qamiltonian tou sust matoc den ekdhl¸nei ˆmesh exˆrthsh apì ton qrìno, h exeligmènh katˆstash grˆfetai

$$|\Psi(t)\rangle = e^{iH(t-t_0)}|\Psi(t_0)\rangle \tag{4.1.2}$$

O telest c $e^{iH(t-t_0)}$ eÐnai o telest c qronik c exèlixhc tou sust matoc, o opoÐoc eÐnai monadiakìc (afoÔ h qamiltonian H eÐnai ènac ermitianìc telest c). AfoÔ o telest c qronik c exèlixhc eÐnai ènac monadiakìc telest c, eÐnai antistrèyimoc. 'Etsi mporoÔme prosdiorÐsoume thn arqik katˆstash dr¸ntac sthn telik me ton antÐstrofo telest qronik c exèlixhc.

Epomènwc eˆn gnwrÐzoume thn katˆstash tou sust matoc mia qronik stigm , tìte ja gnwrÐzoume thn katˆstash tou se opoiad pote qronik stigm (sto mèllon sto pareljìn). H plhroforÐa pou èqoume gia to sÔsthma den mporeÐ na qajeÐ.

4.1.2 Arq thc IsodunamÐac

Me bˆsh thn arq thc isodunamÐac ènac parathrht c stˆsimoc se barutikì pedÐo isodunameÐ topikˆ me ènan epitaqunìmeno parathrht sthn apousÐa barÔthtac. (H epitˆqunsh kajorÐzetai apì thn èntash tou pedÐou barÔthtac sth jèsh tou parathrht . Sto isodÔnamo epitaqunìmeno sÔsthma anaforˆc ton rìlo twn barutik¸n dunˆmewn ton èqoun adraneiakèc dunˆmeic.) Se duo sust mata anaforˆc pou eÐnai isodÔnama, ta fusikˆ fainìmena perigrˆfontai apì touc Ðdiouc nìmouc kai exis¸seic.

4.1.3 Mh klwnopoÐhsh kbantik c plhroforÐac

H kbantik plhroforÐa den mporeÐ na klwnopoihjeÐ. Gia na to doÔme autì ja parousiˆsoume èna parˆdeigma [2].

'Estw ìti èqoume èna swmatÐdio me spin pˆnw $|\uparrow\rangle$. Klwnopoi¸ntac to paÐrnoume

$$|\uparrow\rangle \rightarrow |\uparrow\rangle|\uparrow\rangle \tag{4.1.3}$$

paromowc kai gia èna spin k^tw.

$$|\downarrow\rangle \rightarrow |\downarrow\rangle|\downarrow\rangle \tag{4.1.4}$$

'Estw ìti arqik^ èqoume mia epallhla katast^sewn

$$|\Psi\rangle \rightarrow \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \tag{4.1.5}$$

H klwnopohsh thc kat^stashc dnei

$$\begin{aligned} \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big) &\rightarrow \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big)\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big) \\ &= \frac{1}{2}\Big(|\uparrow\rangle|\uparrow\rangle + |\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\Big) \end{aligned} \tag{4.1.6}$$

'Omwc h kbantomhqanik apaite ìloi oi telestèc na enai grammiko, epomènwc h telik kat^stash ja prèpei na enai mia epallhla twn telik¸n katast^sewn stic exis¸seic 4.1.3 kai 4.1.4

$$\frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big) \rightarrow \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\Big) \tag{4.1.7}$$

Oi dÔo katast^seic den enai sec epomènwc katal goume sto sumpèrasma ìti den mpore na gnei klwnopohsh thc plhroforac (efìson h kbantomhqanik perigr^fetai apì grammikoÔc telestèc).

4.1.4 Arq thc sumplhrwmatikìthtac

Me b^sh thn arq thc sumplhrwmatikìthtac kanènac parathrht c den mpore na antilhfje parabash opoioud pote nìmou thc fÔshc, parìlo pou di^foroi parathrhtèc mpore na perigr^foun diaforetik^ to sÔsthma. H perigraf tou enìc sumplhr¸nei thn perigraf tou ^llou [17].

Gia par^deigma, èstw ènac parathrht c brsketai se meg^lh apìstash apì mia maÔrh trÔpa. Autìc o parathrht c blèpei thn Ôlh pou pèftei sthn maÔrh trÔpa na plhsi^zei asumptwtik^ ton orzonta qwrc na ton diapern^. Sto sÔsthma tou, h aktinobola Hawking proèrqetai apì sked^seic swmatidwn sthn jermik atmìsfaira orzonta. Arqik^ h plhrofora brsketai sugkentrwmènh ston orzonta, kai èpeita metafèretai sthn ekpempìmenh aktinobola. Gia èna parathrht pou pèftei eleÔjera, h Ôlh (kai h plhrofora pou metafèrei) pern^ ton orzonta kai telik^ katal gei sthn anwmala. Epomènwc oi dÔo parathrhtèc diafwnoÔn wc proc to pou brsketai h plhrofora ìmwc kanènac den antilamb^netai k^poia parabash.

Gia na isqÔei h arq thc sumplhrwmatikìthtac prèpei oi bajmoÐ eleujerÐa exwterikˆ tou orÐzonta na mhn eÐnai anexˆrthtoi apì autoÔc sto eswterikì tou [2]. Epomènwc oi diˆforoi telestèc pou perigrˆfoun to sÔsthma den eÐnai topikoÐ.

4.1.5 Arq thc olografÐac

'Estw èna barutikì sÔsthma brÐsketai se qwrÐo me ìgko V. Eˆn to sÔsthma perigrˆfetai me mia topik jewrÐa pedÐou, oi bajmoÐ eleujerÐac tou eÐnai anˆlogoi me ton ìgko. Eˆn prosjèsoume sto sÔsthma arket enèrgeia, h barutik tou aktÐna ja eÐnai arketˆ megˆlh ¸ste na peribˆlei to qwrÐo pou katalambˆnei to sÔsthma, kai ètsi ja katarreÔsei se mia maÔrh trÔpa. Oi bajmoÐ eleujerÐac miac maÔrhc trÔpac eÐnai anˆlogoi me to embadìn tou orÐzonta (se monˆdec Planck). Wc apotèlesma ja eÐqame meÐwsh thc entropÐac kai parabÐash tou deÔterou jermodunamikoÔ nìmou. Epomènwc h ektÐmhsh ìti oi bajmoÐ eleujerÐac eÐnai anˆlogoi tou ìgkou den mporeÐ na isqÔei gia èna barutikì sÔsthma. Oi bajmoÐ eleujerÐac enìc sust matoc eÐnai anˆlogoi me to embadìn thc epifˆneiac pou peribˆllei to qwrÐo [18].

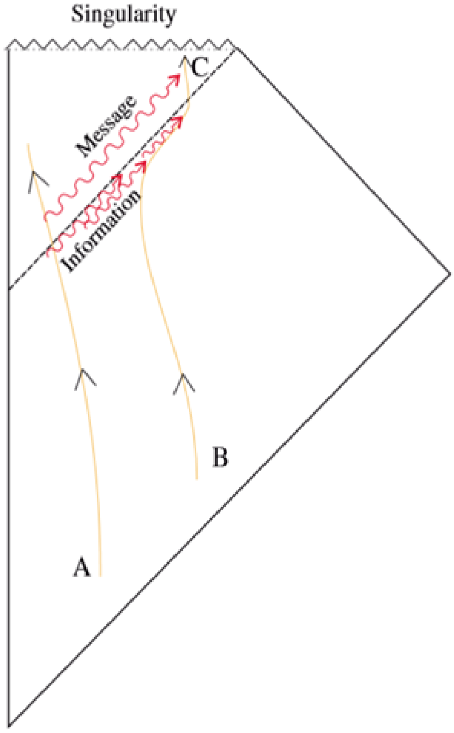
T¸ra eÐnai dunatìn mia topik jewrÐa orismènh sto sÔnoro na perigrˆfei to sÔsthma. Aut h jewrÐa sto sÔnoro (olìgramma) èqei mia diˆstash ligìterh se sqèsh me to barutikì sÔsthma [18].

SÔmfwna loipìn me thn arq thc olografÐac èna barutikì sÔsthma perigrˆfetai apì mia jewrÐa sto sÔnoro tou. Sthn perÐptwsh barutik¸n susthmˆtwn sto qwroqrìno anti de Sitter, h isodÔnamh perigraf eÐnai mia sÔmmorfh jewrÐa pedÐou (qwrÐc barÔthta) sto sÔnoro. O duismìc autìc onomˆzetai antistoiqÐa AdS/CFT [19].

4.2 Parˆdoxo klwnopoÐhshc plhroforÐac

Ac jewr soume dÔo parathrhtèc, ton A pou pèftei eleÔjera se mia maÔrh trÔpa kai ton B pou mènei akÐnhtoc sthn exwterik perioq thc maÔrhc trÔpac. AfoÔ perˆsei ton orÐzonta, o parathrht c A ekpèmpei èna kbˆnto plhroforÐac. To kbˆnto ja katal xei sthn anwmalÐa metˆ apì kˆpoio qronikì diˆsthma. O parathrht c B afoÔ sullèxei thn plhroforÐa apì thn aktinobolÐa Hawking pèftei mèsa sthn maÔrh trÔpa. AfoÔ eisèljei sto eswterikì thc maÔrhc trÔpac ja mporoÔse na anakt sei to arqikì kbˆnto plhroforÐac pou èsteile o parathrht c A, protoÔ autì katal xei sthn anwmalÐa (eikìna 4.2.1). Eˆn ìmwc sumbeÐ autì, o parathrht c B parathreÐ klwnopoÐhsh plhroforÐac. Wc ek toÔtou h sumplhrwmatikìthta den eÐnai mia sunep c jewrÐa, afoÔ o parathrht c B ja antilambˆnetai parabÐash enìc nìmou thc fÔshc.

Ac upologÐsoume to qronikì perij¸rio pou èqei o parathrht c A na steÐlei to s ma (ètsi ¸ste na katafèrei na to pˆrei o B protoÔ katal xei sthn anwmalÐa.



Eikona 4.2.1: Di^gramma Penrose gia touc 2 parathrhtèc. O parathrht c B afoÔ anakt sei to kb^nto plhroforÐac tou A eisèrqetai mèsa sthn maÔrh trÔpa [2].

SÔmfwna me ton Page [8] to pr¸to kb^nto plhroforÐac diafÔgei apì thn maÔrh trÔpa ìtan h entropÐa thc elattwjeÐ sth mis . Epomènwc o parathrht c B prèpei na perimènei arketì qrìno sthn exwterik perioq , mèqri na anakt sei thn plhroforÐa mèsw thc aktinobolÐac. O qrìnoc pou qrei^zetai gia na elattwjeÐ h entropÐa sto misì eÐnai thc t^xhc

$$t^* \sim M^3 G^2 \tag{4.2.1}$$

Ja qrhsimopoi soume tic suntetagmènec Kruskal-Skezeres T, Z oi opoÐec dÐnontai apì tic ekfr^seic

$$Z^2 - T^2 = \frac{GM}{2}(r - 2GM)e^{\frac{r}{2GM}}, \quad \frac{T}{Z} = \tanh\frac{t}{4GM} \tag{4.2.2}$$

O parathrht c A akoloujeÐ thn kosmik gramm $Z_0 =$ staj. kai pern^ apì ton orÐzonta $T = Z$ thn qronik stigm $T_0 = Z_0$. O parathrht c B akoloujeÐ thn kampÔlh $r = 4GM$ mèqri thn qronik stigm $t = t_1$, opìte anakt^ to kb^nto plhroforÐac tou A apì thn aktinobolÐa. Oi suntetagmènec tou gegonìtoc dÐnontai apì tic ekfr^seic

$$Z_1 = GMe\cosh\frac{t_1}{4GM}, \quad T_1 = GMe\sinh\frac{t_1}{4GM} \tag{4.2.3}$$

AkoloÔjwc o parathrht c B pèftei sthn maÔrh trÔpa akolouj¸ntac thn kosmik gramm $Z = Z_1$. Apì tic sqèseic 4.2.1 kai 4.2.3, blèpoume ìti $Z_1 >> \sqrt{2GM}$. O parathrht c B ftˆnei sthn anwmalÐa thn qronik stigm T_2

$$T_2 = \sqrt{Z_1^2 + G^2 M^2} \tag{4.2.4}$$

Gia na katafèrei na pˆrei to s ma o parathrht c B (protoÔ katal xei sthn anwmalÐa), o parathrht c A prèpei na to ekpèmyei entìc tou qronikoÔ diast matoc $[T_0, T_3]$, ìpou $T_0 < T_3 < T_2$. Eˆn o A ekpèmyei to s ma thn qronik stigm T_3, to s ma kai o parathrht c B katal goun sthn anwmalÐa tautìqrona th qronik stigm T_2. To fwteinì s ma akoloujeÐ thn kosmik gramm $T + Z =$staj.. Epomènwc

$$\Delta T = T_3 - T_0 = \sqrt{Z_1^2 + G^2 M^2} - Z_1 \tag{4.2.5}$$

Epeid $Z_1 >> GM$, paÐrnoume

$$\Delta T = \frac{G^2 M^2}{2 Z_1} \tag{4.2.6}$$

Qrhsimopoi¸ntac tic ekfrˆseic 4.2.1 kai 4.2.3, brÐskoume

$$\Delta T \sim \frac{GM}{e} e^{-\frac{M^2 G}{4}} \tag{4.2.7}$$

SÔmfwna me thn arq thc abebaiìthtac

$$\Delta T \Delta E \geq \frac{1}{2} \tag{4.2.8}$$

'Etsi h enèrgeia pou qreiˆzetai o parathrht c A gia na steÐlei to s ma eÐnai

$$\Delta E > \frac{e}{2GM} e^{\frac{M^2 G}{4}} \tag{4.2.9}$$

polÔ megalÔterh apì thn enèrgeia thc maÔrhc trÔpac. Epomènwc o parathrht c B den mporeÐ na lˆbei thn plhroforÐa kai ètsi den parathreÐtai parabÐash kˆpoiou nìmou thc fÔshc.

4.3 Diat rhsh baruonikoÔ arijmoÔ

'Ena ˆllo parˆdoxo sqetÐzetai me thn diat rhsh tou baruonikoÔ arijmoÔ [2]. Ta perissìtera swmatÐdia qarakthrÐzontai apì kbantikoÔc arijmoÔc pou sundèontai me to spin, to fortÐo, to qr¸ma to baruonikì arijmì, k.t.l.. Ta quarks èqoun baruonikì arijmì $+1/3$ kai ta antiquarks $-1/3$. Ta baruìnia, ìpwc ta prwtonia kai ta netrìnia, èqoun baruonikì arijmì +1, kai apoteloÔntai apì 3 quarks. O baruonikìc arijmìc diathreÐtai stic pleÐstec diergasÐec, parìlo pou se merikˆ montèla fusik c pèran tou kajierwmènou protÔpou epitrèpetai h parabÐash tou.

'Estw ìti èqoume èna astèra me m^za merikèc forèc megalÔterh apì thn m^za tou liou. Ston astèra autì up^rqoun perÐpou 10^{57} prwtìnia kai netrìnia me to kajèna na èqei baruonikì arijmì +1. 'Otan exantlhjoÔn ta kaÔsima tou astèra, ja katarreÔsei se mia maÔrh trÔpa h opoÐa ja exaôlwjeÐ se aktinobolÐa Hawking. H jermokrasÐa Hawking miac maÔrhc trÔpac me aut th m^za isoÔtai perÐpou me $10^{-8}eV$ (exÐswsh 2.3.13) kai ètsi den mporoÔn na ekpemfjoÔn baruìnia twn opoÐwn h m^za eÐnai thc t^xhc tou $1GeV$. Epomènwc o baruonikìc arijmìc twn swmatidÐwn pou pèftoun mèsa se mia maÔrh trÔpa q^netai (se meg^lo bajmì).

H mh diat rhsh tou baruonikoÔ arijmoÔ den apoteleÐ par^doxo epeid up^rqoun montèla pou epitrèpoun thn parabÐash tou. Mia tètoia diergasÐa eÐnai h akìloujh

$$p \rightarrow X + e^{+} \tag{4.3.1}$$

Se aut thn diergasÐa èna prwtìnio diasp^tai se èna pozitrìnio kai èna bajmwtì swmatÐdio, to opoÐo sumbolÐzoume me Q. H telik kat^stash èqei baruonikì arijmì 0. Ta prwtìnia eÐnai stajer^ swmatÐdia me meg^lo qrìno zw c, perÐpou 10^{32} qrìnia. Epomènwc h m^za tou swmatidÐou Q prèpei na eÐnai meg^lh ètsi ¸ste h di^spash na èqei mikr pijanìthta na sumbeÐ.

Ac jewr soume loipìn èna prwtìnio pou pèftei mèsa se mia maÔrh trÔpa. To prwtìnio q^nei ton baruonikì arijmì tou mèsw thc pio p^nw di^spashc. Gia na sumbeÐ h di^spash aut prèpei na dojeÐ arket enèrgeia sto prwtìnio ¸ste na paraqjoÔn ta telik^ proðìnta. To par^doxo sundèetai me to pou lamb^nei q¸ra h di^spash. SÔmfwna me ènan parathrht sto ^peiro, h jermokrasÐa kont^ ston orÐzonta eÐnai polÔ meg^lh. Kaj¸c to prwtìnio plhsi^zei ton orÐzonta, se apìstash thc t^xhc tou $1/M_X$, h jermokrasÐa eÐnai thc t^xhc M_X, ìpou M_X h m^za tou swmatidÐou Q. Epomènwc h di^spash mporeÐ na sumbeÐ sthn perioq aut , afoÔ ètsi to prwtìnio mporeÐ na apokt sei thn apaitoÔmenh enèrgeia.

Gia èna parathrht pou pèftei eleÔjera ìmwc, mazÐ me to prwtìnio, h jermokrasÐa eÐnai mhdamin . Gia autìn ton parathrht h di^spash sumbaÐnei polÔ kont^ sthn anwmalÐa, ìpou apokt^ thn apaitoÔmenh enèrgeia exaitÐac thc meg^lhc kampulìthtac tou qwrìqronou.

Oi dÔo parathrhtèc diafwnoÔn sqetik^ me to pou sumbaÐnei h di^spash. H arq thc sumplhrwmatikìthtac epib^llei ìti kai oi dÔo parathrhtèc èqoun dÐkaio!

H pijanìthta na brejeÐ to swmatÐdio se kat^stash me mhdèn baruonikì arijmì den eÐnai amelhtèa. H pijanìthta aut dÐnetai apì thn èkfrash [2]

$$Pi \sim \frac{g^2}{4\pi} \log \frac{\kappa}{M_X} \tag{4.3.2}$$

ìpou g h stajer^ sÔzeuxhc kai κ to ìrio apokop c (cuto□) sthn jewrÐa pedÐwn. Sugkekrimèna e^n $g \sim 1$, to kat¸fli κ na eÐnai thc t^xhc thc m^zac Planck kai to M_X thc t˜xhc $10^{16} GeV$ [1], tìte h pijanìthta eÐnai sugkrÐsimh me thn mon˜da. Gia na eÐnai sumbatì autì me thn stajerìthta tou prwtonÐou, jewroÔme ìti oi metab˜seic metaxÔ twn dÔo katast˜sewn eÐnai polÔ gr gorec. QwrÐc epiprìsjeth enèrgeia, to prwtìnio mporeÐ na metabeÐ sthn kat˜stash mhdèn baruonikoÔ arijmoÔ gia qrìno thc t˜xhc $\frac{1}{M_X}$, sÔmfwna me thn arq thc abebaiìthtac $\Delta t \Delta E \sim 1$. Autèc tic metab˜seic den mporoÔme na tic aniqneÔsoume epeid eÐnai polÔ sÔntomec.

Kaj¸c to prwtìnio plhsi˜zei ton orÐzonta, o exwterikìc parathrht c antilamb˜netai tic metab˜seic autèc na sumbaÐnoun ìlo kai pio argìrujma exaitÐac tou fainomènou thc diastol c tou qrìnou. 'Etsi mporeÐ na tic diakrÐnei tic di˜forec katast˜seic. Gia ton exwterikì parathrht , h jermokrasÐa kont˜ ston orÐzonta eÐnai arket˜ meg˜lh gia na epitrèpontai tètoiec metab˜seic.

Ti sumbaÐnei ìmwc sthn perÐptwsh pou o parathrht c pou pèftei eleÔjera steÐlei s ma ston exwterikì parathrht ìti h di˜spash den sumbaÐnei sth jèsh ìpou exwterikìc parathrht c sumperaÐnei ìti h di˜spash èqei sumbeÐ; O parathrht c pou pèftei eleÔjera prèpei na thn k˜nei thn mètrhsh sthn perioq pou o A antilamb˜netai na sumbaÐnei h di˜spash, prin per˜sei ton orÐzonta. 'Etsi èqei qronikì perij¸rio thc t˜xhc $1/M_X$ gia na ektelèsei thn mètrhsh. 'Omwc gia na metr sei k˜ti tìso gr gora qrei˜zetai enèrgeia thc t˜xhc M_X, pou eÐnai h enèrgeia pou apaiteÐtai ¸ste to prwtìnio na diaspasteÐ. Epomènwc sthn prosp˜jeia tou o parathrht c na prosdiorÐsei thn kat˜stash tou prwtonÐou prokaleÐ o Ðdioc thn di˜spash.

Katal goume sto sumpèrasma ìti kanènac apì touc 2 parathrhtèc den antilamb˜netai parabÐash k˜poiou nìmou thc fÔshc, sÔmfwna me thn arq thc sumplhrwmatikìthtac.

4.4 Par˜doxo diat rhshc plhroforÐac kat˜ thn exaÖlwsh miac maÔrhc trÔpac

4.4.1 Proèleush paradìxou

'Estw ìti èqoume ènan astèra se mia kajar kat˜stash, o opoÐoc katarrèei se mia maÔrh trÔpa. Sthn sunèqeia h maÔrh trÔpa exaôl¸netai. H telik kat˜stash apoteleÐtai apì thn jermik aktinobolÐa Hawking, mia meikt kat˜stash. H jermik aktinobolÐa den metafèrei thn plhroforÐa sqetik˜ me thn arqik kat˜stash ètsi h plhroforÐa q˜netai.

[1] Μια ϑεωρία που επιτρέπει αυτή την διάσπαση του πρωτονίου είναι η ϑεωρία μεγάλης ενοποίησης $SU(5)$, η οποία ενοποιεί τις ασϑενείς και τις ισχυρές αλληλεπιδράσεις. Σε αυτή την ϑεωρία το σωματίδιο X έχει μάζα της τάξης του $10^{16} GeV$, που είναι η κλίμακα μεγάλης ενοποίησης.

4.4.2 Prot^seic gia lÔsh tou paradìxou

Gia maÔrec trÔpec me meg^lh m^za h barÔthta kont^ ston orzonta enai asjen c epomènwc mporoÔme na qrhsimopoi soume thn hmiklasik prosèggish. Autì shmanei ìti mporoÔme na perigr^youme to sÔsthma qrhsimopoi¸ntac thn genik sqetikìthta kai thn kbantik jewra pedou, epomènwc o upologismìc tou Hawking enai arket^ akrib c.

4.4.2.1 Prìtash Page

Mia prìtash gia thn eplush tou probl matoc enai aut tou Page [8]. Me b^sh thn prìtash aut , h aktinobola Hawking metafèrei thn plhrofora. H entropa sÔmplexhc metaxÔ twn bajm¸n eleujerac pou brskontai mèsa sthn maÔrh trÔpa me autoÔc èxw prèpei na akolouje thn kampÔlh Page [8, 2]. Kaj¸c h entropa Bekenstein-Hawking thc maÔrhc trÔpac elatt¸netai, h entropa sÔmplexhc aux^netai. 'Otan h entropa thc maÔrhc trÔpac elattwje perpou sth mis thc arqik c thc tim c, h entropa sÔmplexhc prèpei na arqsei na elatt¸netai, ìmwc gia na elattwje prèpei h plhrofora na arqsei na exèrqetai apì thn maÔrh trÔpa. Autì ìmwc apagoreÔetai apì thn topikìthta kai thn aitiatikìthta. Sta telik^ st^dia thc exaölwsic miac maÔrhc trÔpac ìpou h aktna tou orzonta enai thc t^xhc tou m kouc Planck, swc h pl rhc kbantik perigraf tou sust matoc epitrèpei thn exagwg plhroforac. 'Omwc sto telikì autì st^dio apomènei polÔ lgh plhrofora, thc t^xhc tou embadoÔ tou orzonta se mon^dec Planck, h opoa enai polÔ mikr .

4.4.2.2 Parabash kbantomhqanik c

Mia ^llh prìtash enai ìti h kbantomhqanik den mpore na perigr^yei sust mata me isqur barÔthta. Aut h ^poyh den enai epark¸c tekmhriwmènh kai enai asÔmbath me th jewra twn qord¸n, h opoa apotele mia jewra kbantik c barÔthtac. To par^doxo di-at rhshc thc plhroforac ja mporoÔse na epiluje sth b^sh thc olografik c antistoiqac AdS/CFT [17], pou ikanopoie ìlouc touc nìmouc thc kbantik c mhqanik c.

4.4.2.3 SÔmplexh metaxÔ uperÔjrwn kai uperiwd¸n bajm¸n eleujerac

H jermik aktinobola Hawking mpore na jewrhje wc mia uperi¸dhc (Hart) diergasa, afoÔ to pl^toc ekpomp c thc den parousi^zei ton qarakthristikì pìlo thc upèrujrhc ak-tinobolac [2]. Ta kb^nta Hawking prèpei na sunodeÔontai apì ^peiro arijmì uperÔjrwn swmatidwn me enèrgeia polÔ mikrìterh thc jermokrasac Hawking. Ta upèrujra swmat-dia enai sumplegmèna me ta kb^nta Hawking, kai ètsi mporoÔn na metafèroun mèroc thc plhroforac pou lepei [4, 20].

[2] Στην περίπτωση της ακτινοβολίας Hawking, η ενεργειακή κατανομή των εκπεμπόμενων σωματιδίων ακολουθεί την καμπύλη μελανού σώματος, στην οποία η συνεισφορά των σωματιδίων με ασυμπτωτικά μηδενική ενέργεια μηδενίζεται.

Oi megˆloi metasqhmatismoð bajmðdac (LGT) epibˆlloun thn parousða twn upèrujrwn swmatidðwn (fwtìnia kai barutìnia) sthn telik katˆstash, h opoða upojètoume ìti eðnai mia kajar katˆstash [4]. To uperi¸dec mèroc thc apoteleðtai apì thn aktinobolða Hawking kai to upèrujro mèroc apoteleðtai apì ta upèrujra swmatðdia. Paðrnontac to merikì ðqnoc wc proc ìlouc touc upèrujrouc bajmoÔc eleujerðac, katal goume ston jermikì pðnaka puknìthtac tou Hawking.

To mètro thc plhroforðac pou metafèroun ta upèrujra swmatðdia dðdetai apì thn entropða sÔmplexhc. Sthn ergasða [1] èqei upologisteð diataraktikˆ h kÔria tˆxh thc entropðac sÔmplexhc metaxÔ uperÔjrwn kai uperiwd¸n bajm¸n eleujerðac se skedˆseic dÔo hlektronðwn, kai èqei brejeð ìti apoklðnei logarijmikˆ ìtan to upèrujro ìrio apokop c teðnei sto mhdèn. H entropða sÔmplexhc eðnai mh mhdenik kai apoklðnei se ìlec tic tˆxeic sth jewrða diataraq¸n. Sto deÔtero mèroc (mèroc II) thc ergasðac aut c upologðzoume thn entropða sÔmplexhc pr¸thc kai deÔterhc tˆxhc, gia aujaðretec skedˆseic sthn QED. Parìmoia sumperˆsmata prokÔptoun kai sth barÔthta kai genikˆ se ìlec tic jewrðec bajmðdac.

Ac exetˆsoume pio prosektikˆ thn idèa aut [4]. JewroÔme mia maÔrh trÔpa pou dhmiourg jhke mèsw barutik c katˆrreushc, kai sth sunèqeia exaôl¸netai. H pijanìthta anˆ monˆda qrìnou na ekpemfjeð èna kbˆnto Hawking me suqnìthta ω eðnai anˆlogh thc èkfrashc

$$N(\omega) \sim \frac{\omega^2}{e^{\omega/T_H} - 1} \tag{4.4.1}$$

ìpou $T_H = 1/8\pi GM$ h jermokrasða Hawking. ParathroÔme ìti h pijanìthta aut mhdenðzetai kaj¸c $\omega \to 0$. SÔmfwna me ton Hawking h telik katˆstash perigrˆfetai apì èna jermikì pðnaka puknìthtac thc morf c

$$|\Psi\rangle_{in} = \rho_{Hawking} = \sum_\alpha \rho_\alpha |H_\alpha\rangle \langle H_\alpha| \tag{4.4.2}$$

me touc telestèc ρ_α na dðdontai apì thn exðswsh 4.4.1.

Ac upojèsoume ìti èqoume mia monadiak diergasða

$$|\Psi\rangle_{out} = S|\Psi\rangle_{in} = \sum_\alpha c_\alpha |H_\alpha\rangle \tag{4.4.3}$$

ìpou c_α stajeroð suntelestèc. Tètoiec katastˆseic ìmwc, me peperasmèno arijmì uperÔjrwn swmatidðwn, den epitrèpontai, afoÔ to antðstoiqo plˆtoc metˆbashc mhdenðzetai [20]. Gia na èqoume peperasmèno plˆtoc metˆbashc prèpei ta uperi¸dh swmatðdia na sunodeÔontai apì nèfh me ˆpeira upèrujra fwtìnia kai barutìnia [16], me enèrgeia mikrìterh apì $E << T_H$. 'Etsi h sqèsh 4.4.3 paÐrnei thn morf

$$|\Psi\rangle_{out} = S\,|\Psi\rangle_{in} = \sum_{\alpha} c_\alpha\,|H_\alpha\rangle\,|S_\alpha\rangle \tag{4.4.4}$$

ìpou èqoume qwrÐsei ton q¸ro Hilbert se upèrujro kai uperi¸dec mèroc. 'Omwc an kai gnwrÐzoume ìti upèrujra swmatÐdia prèpei na up^rqoun (me b^sh touc nìmouc diat rhshc pou sundèontai me touc meg^louc metasqhmatismoÔc bajmÐdac), den gnwrÐzoume pwc na upologÐsoume akrib¸c to f^sma touc.

'Estw ìti oi upèrujrec katast^seic eÐnai orjokanonikèc

$$\langle S_\beta|S_\alpha\rangle = \delta_{\beta\alpha} \tag{4.4.5}$$

PaÐrnontac to merikì Ðqnoc wc proc touc upèrujrouc bajmoÔc eleujerÐac, prokÔptei ènac pÐnakac puknìthtac thc morf c

$$\rho_H = \sum_{\alpha} c_\alpha c_\alpha^*\,|H_\alpha\rangle\,\langle H_\alpha| \tag{4.4.6}$$

E^n oi suntelestèc c_α ikanopoioÔn th sqèsh

$$c_\alpha = \sqrt{\rho_\alpha}\,e^{i\theta_\alpha} \tag{4.4.7}$$

tìte prokÔptei o jermikìc pÐnakac Hawking

$$\rho_H = \rho_{\text{Hawking}} \tag{4.4.8}$$

'Ara loipìn

$$|\Psi\rangle_{out} = \sum_{\alpha} \sqrt{\rho_\alpha}\,e^{i\theta_\alpha}\,|H_\alpha\rangle\,|S_\alpha\rangle \tag{4.4.9}$$

ìpou to pl^toc ρ_α dÐnetai apì thn èkfrash 4.4.2 kai θ_α mia aprosdiìristh f^sh.

Me b^sh thn idèa aut , h exaölwsh miac maÔrhc trÔpac perigr^fetai apì mia monadiak diergasÐa, thc morf c 4.4.3, me to mètro twn suntelest¸n c_α na kajorÐzetai apì thn èkfrash 4.4.7 sthn hmiklassik prosèggish. Oi aniqneutèc diakrÐnoun mìno thn aktinobolÐa Hawking, kai den up^rqei susqètish/sÔmplexh metaxÔ swmatidÐwn twn arqik¸n kai telik¸n kb^ntwn Hawking (ta opoÐa ekpèmpontai prin kai met^ ton qrìno Page).

Mèroc II

EntropÐa sÔmplexhc se sked^seic fortismènwn swmatidÐwn

Eisagwg

Di^forec asumptwtikèc summetrec apagoreÔoun sked^seic pou perilamb^noun mìno uperi¸dh (hard) swmatdia sthn QED kai sth barÔthta [4, 21]. Ta asumptwtik^ swmatdia (dhlad swmatdia pou èqoun swst sumperifor^ sto ^peiro) prèpei na sunodeÔontai apì nèfh upèrujrwn (soft) fwtonwn barutonwn, pèran apì thn ekpempìmenh aktinobola. Ja kaloÔme ta swmatdia aut^ "ntumèna" (dressed). Ta upèrujra kai uperi¸dh swmatdia enai se sÔmplexh metaxÔ touc me apotèlesma ta upèrujra swmatdia na metafèroun plhrofora. Epeid h diakritìthta twn aniqneut¸n enai periorismènh, ènac ^peiroc arijmìc upèrujrwn swmatidwn diafeÔgoun thc anqneushc.

Ja melet soume sked^seic metaxÔ ntumènwn swmatidwn kai ja broÔme thn entropa sÔmplexhc metaxÔ twn uperÔjrwn kai uperiwd¸n bajm¸n eleujerac sthn telik kat^stash. 'Opwc èqoume anafèrei kat^ thn exaÔlwsh miac maÔrhc trÔpac [20], h aktinobola Hawking prèpei na sunodeÔetai kai apì aktinobola upèrujrwn swmatidwn, h opoa pijanìn na metafèrei thn plhrofora pou leipei.

Parnontac to merikì qnoc wc proc touc upèrujrouc bajmoÔc eleujerac sthn telik kat^stash met^ apì skèdash (me ntumèna swmatdia), prokÔptei mia meikt kat^stash. Sunep¸c up^rqei sÔmplexh metaxÔ twn upèrujrwn kai twn uperiwd¸n bajm¸n eleujerac [22, 23]. Ja melet soume sked^seic metaxÔ fortismènwn swmatidwn sthn QED kai ja upologsoume diataraktik^ tic pr¸tec t^xeic thc entropac sÔmplexhc. Gia na apofÔgoume upèrujrouc apeirismoÔc ja epib^loume kat¸tero ìrio λ stic enèrgeiec pou mporoÔn na èqoun ta upèrujra swmatdia. Autì to petuqanoume b^zontac to sÔsthma se èna meg^lo kouti diast^sewn $L \sim 1/\lambda$. Sto tèloc ja p^roume to suneqèc ìrio $\lambda \to 0$. Ja exet^soume epshc sumbatikèc sked^seic sth b^sh Fock, kai tic diaforèc me sked^seic asumptwtik¸n swmatidwn Faddeev-Kulish [24, 16]. Sthn perptwsh sumbatik¸n sked^sewn, ja upologsoume epshc thn deÔterh t^xh thc entropac sÔmplexhc kai ja doÔme pwc sumperifèrontai oi epìmenec t^xeic.

Gia na qwrsoume ton q¸ro Hilbert se upèrujro kai uperi¸dec mèroc ja qreiastoÔme mia enèrgeia anafor^c E pou thn orzoume na enai mikrìterh apì thn m^za hremac tou hlektronou kai kajorzetai apì thn diakritik ikanìthta twn aniqneut¸n. To upèrujro mèroc perilamb^nei ìla ta swmatdia me sunolik enèrgeia mikrìterh apì E kai to uperi¸dec mèroc perilamb^nei swmatdia me sunolik enèrgeia megalÔterh apì E.

'Opwc ja doÔme h entropÐa sÔmplexhc parousi^zei logarijmikì upèrujro apeirismì. H kÔria t^xh eÐnai an^logh tou tetrag¸nou tou pl^touc skèdashc gia ekpomp enìc upèrujrou fwtonÐou me enèrgeia $\lambda < \omega_\gamma < E$ (sto epÐpedo diagramm^twn dèntrou). Parìlo pou o pÐnakac skèdashc Faddeev - Kulish den parousi^zei upèrujrouc apeirismoÔc sthn jewrÐa diataraq¸n [24, 16], h entropÐa sÔmplexhc apoklÐnei logarijmik^ exaitÐac twn ne-f¸n twn upèrujrwn fwtonÐwn pou sunodeÔoun ta fortismèna swmatÐdia. H kÔria t^xh thc entropÐac sÔmplexhc eÐnai èna kl^sma thc mègisthc dunat c tim c thc pou kajorÐzetai apì th diastatikìthta tou q¸rou Hilbert twn katast^sewn enìc upèrujrou fwtonÐou.

'Opwc anafèrame prohgoumènwc, oi diergasÐec skèdashc sthn QED prèpei na ikanopoioÔn èna ^peiro arijmì apì nìmouc diat rhshc pou sundèontai me meg^louc metasqhmatismoÔc bajmÐdac (LGT) [4, 21]. Oi meg^loi metasqhmatismoÐ bajmÐdac den mhdenÐzontai sto ^peiro, all^ ekdhl¸noun gwniak ex^rthsh. Metab^seic metaxÔ sumbatik¸n katast^sewn Fock, oi opoÐec perilamb^noun èna peperasmèno arijmì fwtonÐwn sthn arqik kai telik kat^s-tash, den mporoÔn na ikanopoi soun touc nìmouc diat rhshc pou sqetÐzontai me touc LGT. Wc apotèlesma, ta antÐstoiqa pl^th skèdashc mhdenÐzontai [1]. Gia na ikanopoioÔntai oi nìmoi diat rhshc prèpei na up^rqei ^peiroc arijmìc fwtonÐwn sthn telik kat^stash. O mhdenismìc twn stoiqeÐwn tou pÐnaka S sthn perÐptwsh twn sumbatik¸n katast^sewn Fock sun jwc apodÐdetai sthn ekjetopoÐhsh twn logarijmik¸n upèrujrwn apeirism¸n ex-aitÐac twn eikonik¸n fwtonÐwn [12], ìmwc mporeÐ na ermhneuteÐ kai wc sunèpeia twn ^peirwn nìmwn diat rhshc.

Fusikèc asumptwtikèc katast^seic mporoÔn na kataskeuastoÔn ntÔnontac ta for-tismèna swmatÐdia Fock me èna nèfoc upèrujrwn fwtonÐwn [16]. Ta antÐstoiqa stoiqeÐa tou pÐnaka S eÐnai mh mhdenik^ kai den parousi^zoun upèrujrouc apeirismoÔc [16, 24]. Ta nèfh upèrujrwn fwtonÐwn twn katast^sewn Faddeev-Kulish (FK) kajistoÔn ta fortÐa pou sundèontai me touc LGT anex^rthta apì thn orm tou gumnoÔ fortismènou swmatidÐou [25]. Ta LGT fortÐa exart¸ntai mìno apì to sunolikì fortÐo tou gumnoÔ swmatidÐou, kai ètsi oi nìmoi diat rhshc ikanopoioÔntai [21, 25]. Epomènwc k^je skèdash sthn QED odhgeÐ se mia telik kat^stash me ^peiro arijmì upèrujrwn fwtonÐwn. Stìqoc mac eÐnai na melet soume thn sÔmplexh metaxÔ uperÔjrwn kai uperiwd¸n swmatidÐwn se mia tupik skèdash kai na upologÐsoume thn entropÐa sÔmplexhc diataraktik^.

Arqik^ ja qrhsimopoi soume to ìrio $E < m_e$ gia na diaqwrÐsoume ton q¸ro Hilbert se upèrujro kai uperi¸dec mèroc

$$\mathcal{H} = \mathcal{H}_H \times \mathcal{H}_S$$

ìpou $\mathcal{H}_H$ perilamb^nei katast^seic swmatidÐwn me enèrgeia megalÔterh apì E kai $\mathcal{H}_S$ per-ilamb^nei upèrujra fwtìnia me enèrgeia mikrìterh apì E. Wc arqik kat^stash ja p^roume mia kajar kat^stash pou perilamb^nei èna aujaÐreto sunduasmì apì fortismèna swmatÐ-dia, en¸ h telik kat^stash ja eÐnai mia sumplegmènh kat^stash tou q¸rou $\mathcal{H}_H \times \mathcal{H}_S$, ìpwc kajorÐzetai apì ton pÐnaka skèdashc S. Ja qeiristoÔme diaforetik^ ta upèrujra fwtìnia pou brÐskontai sta nèfh pou sunodeÔoun ta fortismèna swmatÐdia apì epiprìsjeta upèrujra swmatÐdia pou ekpèmpontai kat˘ thn diadikasÐa skèdashc [1].

Stouc upologismoÔc pou akoloujoÔn ja qrhsimopoi soume merikèc qarakthristikèc energeiakèc klÐmakec. 'Opwc dh èqoume anafèrei, h enèrgeia E qwrÐzei ton q¸ro Hilbert se upèrujro kai uperi¸dec mèroc. To kat¸ fli apokop c λ epib˘lletai gia na omalopoihjoÔn upèrujroi apeirismoÐ. H enèrgeia E_d qarakthrÐzei thn enèrgeia twn fwtonÐwn sta nèfh, kai ikanopoieÐ thn anisìthta $\lambda < E_d < E$. H enèrgeia E_d tÐjetai na eÐnai aujaÐreta mikr , ètsi ¸ste na mporoÔme na qrhsimopoi soume ta upèrujra jewr mata gia na aplopoi soume di˘forec ekfr˘seic. Sto tèloc ja p˘roume to ìrio $\lambda \to 0$ krat¸ntac to phlÐko E_d/E stajerì.

Kef^laio 5

Ntumènec katast^seic

Arqik^ ja melet soume tic katast^seic Faddeev-Kulish kai merikèc idiìthtec touc pou ja qreiastoÔn gia ton upologismì thc entropÐac sÔmplexhc.

5.1 Sumb^seic kai sumbolismìc

Ergazìmaste sthn bajmÐda Lorenz, $\partial_\mu A^\mu = 0$, me to dunamikì na ikanopoieÐ thn exÐswsh kÔmatoc $\partial_\nu \partial^\nu A^\mu = 0$. EpÐshc ja qrhsimopoi soume th metrik Minkowski, me thn qronik thc sunist¸sa na eÐnai arnhtik .

Sunart sei telest¸n dhmiourgÐac kai katastrof c to pedÐo gr^fetai

$$A^\mu = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{\omega_{\vec{k}}}} \sum_r \left(\epsilon_r^\mu(\vec{k}) a_r(\vec{k}) e^{ikx} + \epsilon_r^{\mu *}(\vec{k}) a_r^\dagger(\vec{k}) e^{-ikx} \right) \tag{5.1.1}$$

Oi sqèseic orjokanonikìthtac tou dianÔsmatoc pìlwshc $\epsilon_r^\mu(\vec{k})$ eÐnai

$$\epsilon_{r\mu}(\vec{k}) \epsilon_s^{\mu *}(\vec{k}) = \zeta_r \delta_{rs}, \quad \sum_r \zeta_r \epsilon_r^\mu(\vec{k}) \epsilon_r^{\nu *}(\vec{k}) = \eta^{\mu\nu} \tag{5.1.2}$$

ìpou $\zeta_0 = -1,\ \zeta_1 = \zeta_2 = \zeta_3 = 1$. Oi telestèc dhmiourgÐac kai katastrof c twn fwtonÐwn ikanopoioÔn tic akìloujec sqèseic met^jeshc

$$[a_r(\vec{p}), a_s^\dagger(\vec{q})] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \zeta_r \delta_{rs} \tag{5.1.3}$$

O telest c $a_r^\dagger(\vec{p})$ dhmiourgeÐ fwtìnia me orm $\vec{p}$ kai polikìthta $\epsilon_r^\mu(\vec{p})$ kai o telest c $a_r(\vec{p})$ katastrèfei fwtìnia me orm $\vec{p}$ kai polikìthta $\epsilon_r^\mu(\vec{p})$. Oi fusikèc katast^seic prèpei na ikanopoioÔn thn sunj kh Gupta-Bleuler

$$[a_0(\vec{p}), a_3(\vec{p})] \left|\Psi\right\rangle = 0 \tag{5.1.4}$$

Tèloc oi telestèc dhmiourgÐac-katastrof c hlektronÐwn kai pozitronÐwn ikanopoioÔn tic akìloujec sqèseic

$$\{b_r(\vec{p}), b_s^\dagger(\vec{q})\} = \{d_r(\vec{p}), d_s^\dagger(\vec{q})\} = (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta_{rs} \tag{5.1.5}$$

me tou upìloipouc antimetajètec Ðsouc me mhdèn.

5.2 Katast^seic Faddeev-Kulish

Mia gumn kat^stash "ntÔnetai" ìtan dr^sei se aut n o telest c Faddeev-Kulish (FK) e^{R_f} ìpou

$$R_f = \int \frac{d^3\vec{p}}{(2\pi)^3}\hat{\rho}(\vec{p}) \int_\lambda^{E_d} \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}\left(f(\vec{k},\vec{p})\cdot a^\dagger(\vec{k}) - h.c.\right) \tag{5.2.1}$$

ìpou

$$\hat{\rho}(\vec{p}) = \sum_s \left(b^{s\dagger}(\vec{p})b^s(\vec{p}) - d^{s\dagger}(\vec{p})d^s(\vec{p})\right) \tag{5.2.2}$$

o telest c puknìthtac hlektrikoÔ fortÐou. EpÐshc

$$f(\vec{k},\vec{p})\cdot a^\dagger(\vec{k}) = \sum_r f^\mu(\vec{k},\vec{p})\epsilon^*_{r\mu}(\vec{p})a_r^\dagger(\vec{k}) \tag{5.2.3}$$

$$f^\mu(\vec{k},\vec{p}) = e\left(\frac{p^\mu}{pk} - c_\mu\right)e^{-\frac{ipkt_0}{p^0}}, \quad c^\mu = \left(-\frac{1}{2k_0}, \frac{\vec{k}}{2(k_0)^2}\right) \tag{5.2.4}$$

O telest c FK eÐnai monadiakìc. H sun^rthsh $f^\mu(\vec{k},\vec{p})$ apoklÐnei sto ìrio $\vec{k}\to 0$. Stouc akìloujouc upologismoÔc krat^me to ìrio apokop c λ peperasmèno, paÐrnontac to ìrio $\lambda\to 0$ sto tèloc. O qrìnoc anafor^c t_0 mporeÐ na tejeÐ Ðsoc me mhdèn. To tetradi^nusma c^μ èqei mhdenikì mètro: c^2 =0. EpÐshc isqÔei h sqèsh $ck = 1$. ExaitÐac thc sqèshc aut c, h sun^rthsh $f^\mu(\vec{k},\vec{p})$ eÐnai egk^rsia $fk = 0$. To p^nw ìrio tou oloklhr¸matoc sthn exÐswsh 5.2.1 diasfalÐzei ìti mìno fwtìnia me enèrgeia mikrìterh apì E_d emfanÐzontai sto nèfoc pou sunodeÔei ta fortismèna swmatÐdia.

5.3 Ntumèna hlektrìnia

Ac doÔme pwc ntÔnetai èna hlektrìnio me b^sh to metasqhmatismì FK. JewroÔme thn akìloujh kat^stash (gumnoÔ) hlektronÐou sth b^sh Fock

$$|\vec{p},s\rangle = \sqrt{2E_{\vec{p}}}\,b^{s\dagger}(\vec{p})|0\rangle \tag{5.3.1}$$

H antÐstoiqh ntumènh kat^stash paÐrnei thn morf

$$|\vec{p}, s\rangle_{dressed} = |\vec{p}, s\rangle \times e^{\int_{\lambda}^{E_d} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{(2\omega_{\vec{k}})^{1/2}} \left(f(\vec{k},\vec{p})\cdot a^{\dagger}(\vec{k}) - h.c.\right)} |0\rangle \tag{5.3.2}$$

To fortismèno hlektrìnio sunodeÔetai apì èna nèfoc fwtonÐwn, to opoÐo perigr^fetai apì mia kanonikopoihmènh kat^stash. Gia mh mhdenikì λ, h kat^stash tou nèfouc mporeÐ na grafteÐ sthn morf

$$|f_{\vec{p}}\rangle = N_{\vec{p}}\, e^{\int_{\lambda}^{E_d} \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{(2\omega_{\vec{k}})^{1/2}} f(\vec{k},\vec{p})\cdot a^{\dagger}(\vec{k})} |0\rangle \tag{5.3.3}$$

O par^gontac kanonikopoÐhshc dÐnetai apì thn èkfrash

$$N_{\vec{p}} = e^{-\frac{1}{2}\int_{\lambda}^{E_d} \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} f^{\mu}(\vec{q},\vec{p}) f^{*}_{\mu}(\vec{q},\vec{p})} \tag{5.3.4}$$

UpologÐzoume ton ekjèth kai paÐrnoume

$$\frac{1}{2}\int_{\lambda}^{E_d} \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} f^{\mu}(\vec{q},\vec{p}) f^{*}_{\mu}(\vec{q},\vec{p}) = \frac{e^2}{8\pi^2} \log\left(\frac{E_d}{\lambda}\right) I(v) \tag{5.3.5}$$

ìpou $v = |\vec{p}|/p^0$ h taqÔthta tou hlektronÐou kai

$$I(v) = -2 + \frac{1}{v}\log\left(\frac{1+v}{1-v}\right) \tag{5.3.6}$$

mia mh arnhtik sun^rthsh. Sto ìrio ìpou $v \to 1$, h sun^rthsh aut apeirÐzetai logarijmik^. OrÐzontac

$$A_{\vec{p}} = \frac{e^2}{8\pi^2} I(v) \tag{5.3.7}$$

prokÔptei

$$N_{\vec{p}} = \left(\frac{\lambda}{E_d}\right)^{A_{\vec{p}}} \tag{5.3.8}$$

O par^gontac $N_{\vec{p}}$ mhdenÐzetai sto ìrio $\lambda \to 0$. Sunep¸c h exÐswsh 5.3.3 den mporeÐ na qrhsimopoihjeÐ se aut thn perÐptwsh.

5.3.1 Fusikèc idiìthtec nèfouc

O mèsoc arijmìc twn fwtonÐwn sthn kat^stash tou nèfouc dÐdetai apì thn èkfrash

$$\langle f_{\vec{p}}|N_{ph}|f_{\vec{p}}\rangle = \int_{\lambda}^{E_d} \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{2\omega_{\vec{q}}} f^{\mu}(\vec{q},\vec{p}) f^{*}_{\mu}(\vec{q},\vec{p}) = \frac{e^2}{4\pi^2}\log\left(\frac{E_d}{\lambda}\right) I(v) \tag{5.3.9}$$

O arijmìc twn fwtonÐwn apeirÐzetai sto ìrio ìpou $\lambda \to 0$ (ìpwc anafèrame prohgoumènwc). H sunolik enèrgeia pou metafèroun ta fwtìnia dÐnetai apì thn èkfrash

$$\langle f_{\vec{p}}|H_{ph}|f_{\vec{p}}\rangle = \frac{1}{2}\int_{\lambda}^{E_d} \frac{d^3\vec{q}}{(2\pi)^3} f^{\mu}(\vec{q},\vec{p}) f^{*}_{\mu}(\vec{q},\vec{p}) = \frac{e^2}{4\pi^2}(E_d - \lambda) I(v) \tag{5.3.10}$$

Gia tupikèc timèc thc taqÔthtac tou hlektronÐou h enèrgeia eÐnai thc t^xhc E_d. Epomènwc to nèfoc fwtonÐwn an kei sto upèrujro mèroc tou q¸rou Hilbert $\mathcal{H}_s$. H orm tou nèfouc dÐnetai apì thn èkfrash

$$\langle f_{\vec{p}}|\vec{P}_{ph}|f_{\vec{p}}\rangle = \int_{\lambda}^{E_d} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\vec{q}}{2\omega_{\vec{q}}} f^{\mu}(\vec{q},\vec{p}) f^{*}_{\mu}(\vec{q},\vec{p}) = \frac{e^2}{8\pi^2}(E_d - \lambda)\left(\frac{3}{v}I(v) - vI(v) - 2V\right)\hat{p} \tag{5.3.11}$$

Sto ìrio ìpou to hlektrìnio plhsi^zei thn taqÔthta tou fwtìc, h enèrgeia kai h orm tou nèfouc apoklÐnoun logarijmik^ kai gÐnontai Ðsec. H enèrgeia kai orm tou nèfouc paramènoun polÔ mikrìterec apì thn enèrgeia kai thn orm tou hlektronÐou.

Ja upologÐsoume t¸ra thn anamenìmenh tim tou dunamikoÔ sthn kat^stash tou nèfouc. Aut eÐnai

$$\bar{A}_{\mu}(x) = \langle f_{\vec{p}}|A_{\mu}(x)|f_{\vec{p}}\rangle = \int_{\lambda}^{E_d} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\vec{q}}{2\omega_{\vec{q}}} \left(f_{\mu}(\vec{q},\vec{p})e^{iqx} + f^{*}_{\mu}(\vec{q},\vec{p})e^{-iqx}\right) \tag{5.3.12}$$

Epeid h sun^rthsh $f_{\mu}(\vec{q},\vec{p})$ eÐnai egk^rsia, to $\bar{A}_{\mu}(x)$ ikanopoieÐ thn sunj kh Lorenz kai thn kumatik exÐswsh.

5.4 Genikèc katast^seic

Ac exet^soume pwc ntÔnoume mia genik kat^stash hlektronÐwn, pozitronÐwn kai fwtonÐwn, $\alpha = \{e_i, \vec{p}_i, s_i\}$. H kat^stash tou nèfouc $|f_{\alpha}\rangle$ dÐnetai apì tic ekfr^seic 5.3.2 kai 5.3.3, e^n h sun^rthsh $f_{\mu}(\vec{q},\vec{p})$ antikatastajeÐ me thn akìloujh

$$f^{\mu}_{\alpha}(\vec{k}) = \sum_{i\in\alpha} e_i \left(\frac{p^{\mu}_i}{p_i k} - c^{\mu}\right) e^{-ip_i k t_0/p^0_i} \tag{5.4.1}$$

ìpou e_i to fortÐo kai p_i h orm tou swmatidÐou i. Gia ta fwtìnia den èqoume suneisfor^ epeid èqoun mhdenikì fortÐo. Gia touc upologismoÔc pou akolouj oÔn, jètoume $t_0 = 0$.

O par^gontac kanonikopoÐhshc N_{α} paÐrnei thn morf thc exÐswshc 5.3.4 me ekjèth

$$\begin{aligned}\int_{\lambda}^{E_d} \frac{d^3\vec{q}}{2(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} f^{\mu}_{\alpha}(\vec{q}) f^{*}_{\alpha\mu}(\vec{q}) &= \int_{\lambda}^{E_d} \frac{d^3\vec{q}}{2(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} \sum_{ij\in\alpha} e_i e_j \left(\frac{p_i p_j}{(p_i q)(p_j q)} - \frac{c p_i}{p_i q} - \frac{c p_j}{p_j q}\right) \\ &= \frac{1}{8\pi^2}\log\left(\frac{E_d}{\lambda}\right)\sum_{ij\in\alpha} e_i e_j I(\vec{v}_i, \vec{v}_j)\end{aligned} \tag{5.4.2}$$

ìpou

$$I(\vec{v_i},\vec{v_j}) = \frac{1}{2v_i}\log\left(\frac{1+v_i}{1-v_i}\right) + \frac{1}{2v_j}\log\left(\frac{1+v_j}{1-v_j}\right) - \frac{1}{2v_{ij}}\log\left(\frac{1+v_{ij}}{1-v_{ij}}\right) - 1 \tag{5.4.3}$$

kai v_{ij} eÐnai to mètro thc sqetik c taqÔthtac metaxÔ twn swmatidÐwn i kai j:

$$v_{ij} = \left(1 - \frac{m_i^2 m_j^2}{(p_i \cdot p_j)^2}\right)^{\frac{1}{2}} \tag{5.4.4}$$

Oi diag¸nioi ìroi $i = j$ anapar^goun thn èkfrash 5.3.6. OrÐzontac

$$\mathcal{A}_\alpha = \frac{1}{8\pi^2}\sum_{ij\in\alpha} e_i e_j I(\vec{v_i},\vec{v_j}) \tag{5.4.5}$$

paÐrnoume

$$N_\alpha = \left(\frac{\lambda}{E_d}\right)^{\mathcal{A}_\alpha} \tag{5.4.6}$$

Sth genikìterh perÐptwsh, o ekjèthc A_α eÐnai jetikìc kai mh mhdenikìc, kai epomènwc o par^gontac N_α mhdenÐzetai gia $\lambda \to 0$.

'Allh mia sqèsh pou ja qreiasteÐ sthn sunèqeia eÐnai to pl^toc allhlepik^luyhc metaxÔ dÔo katast^sewn pou perigr^foun nèfh

$$\langle f_\beta | f_\alpha \rangle = N_\beta N_\alpha e^{\int_\lambda^{E_d} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} f_\alpha^\mu(\vec{q}) f_{\beta\mu}^*(\vec{q})} \tag{5.4.7}$$

O ekjèthc dÐnetai apì thn èkfrash

$$\begin{aligned}\int_\lambda^{E_d} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} f_\beta^\mu(\vec{q}) f_{\alpha\mu}^*(\vec{q}) &= \int_\lambda^{E_d} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} \sum_{i\in\alpha}\sum_{i\in\beta} e_i e_j \left(\frac{p_i p_j}{(p_i q)(p_j q)} - \frac{c p_i}{p_i q} - \frac{c p_j}{p_j q}\right) \\ &= \frac{1}{4\pi^2}\log\left(\frac{E_d}{\lambda}\right)\sum_{i\in\alpha}\sum_{i\in\beta} e_i e_j I(\vec{v_i},\vec{v_j}) \end{aligned} \tag{5.4.8}$$

Jètontac

$$\mathcal{A}_{\alpha\beta} = \frac{1}{4\pi^2}\sum_{i\in\alpha}\sum_{i\in\beta} e_i e_j I(\vec{v_i},\vec{v_j}) \tag{5.4.9}$$

paÐrnoume

$$\langle f_\beta | f_\alpha \rangle = \left(\frac{\lambda}{E_d}\right)^{\mathcal{A}_\alpha + \mathcal{A}_\beta - \mathcal{A}_{\alpha\beta}} \tag{5.4.10}$$

Sthn perÐptwsh ìpou to sunolikì fortÐo twn katast ˆ sewn eÐnai Ðso, $Q_\alpha = Q_\beta$, paÐrnoume

$$\mathcal{A}_\alpha + \mathcal{A}_\beta - \mathcal{A}_{\alpha\beta} = \mathcal{B}_{\alpha\beta} = -\frac{1}{16\pi^2}\left(\sum_{ij\in\alpha} + \sum_{i\in\beta} - 2\sum_{i\in\alpha}\sum_{i\in\beta}\right) e_i e_j \frac{1}{v_{ij}} \log\left(\frac{1+v_{ij}}{1-v_{ij}}\right) \tag{5.4.11}$$

JewroÔme ta swmatÐdia thc kat ˆ stashc β wc exerqìmena kai aut ˆ thc kat ˆ stashc α wc eiserqìmena. O pio p ˆ nw suntelest c gr ˆ fetai

$$\mathcal{B}_{\alpha\beta} = -\frac{1}{16\pi^2}\sum_{ij}\eta_i\eta_j e_i e_j \frac{1}{v_{ij}}\log\left(\frac{1+v_{ij}}{1-v_{ij}}\right) \tag{5.4.12}$$

ìpou orÐsame to η_i na paÐrnei tim $+1$ gia exerqìmena swmatÐdia kai -1 gia eiserqìmena swmatÐdia. AjroÐzoume wc proc ìla ta eiserqìmena kai exerqìmena swmatÐdia. O suntelest c $\mathcal{B}_{\alpha\beta}$ eÐnai jetikìc kai ètsi to pl ˆ toc allhlepik ˆ luyhc mhdenÐzetai sto ìrio $\lambda \to 0$ (se ìlec tic t ˆ xeic wc proc to fortÐo tou hlektronÐou). H telik èkfrash èqei thn morf

$$\langle f_\beta | f_\alpha \rangle = \left(\frac{\lambda}{E_d}\right)^{\mathcal{B}_{\alpha\beta}} \tag{5.4.13}$$

5.5 PÐnakac skèdashc Faddeev-Kulish

Arqik ˆ jamelet soume sked ˆ seic $\alpha \to \beta$ qwrÐc upèrujra fwtìnia enèrgeiac mikrìterhc apì E_d sthn arqik kai telik kat ˆ stash. Ja upologÐsoume ta stoiqeÐa tou pÐnaka S gia tic ntumènec katast ˆ seic, sunart sei twn sumbatik¸n pinakostoiqeÐwn sth b ˆ sh Fock [24]:

$$\tilde{S}_{\alpha\beta} ={}_d \langle\alpha|S|\beta\rangle_d \tag{5.5.1}$$

$$S_{\alpha\beta} = \langle\alpha|S|\beta\rangle \tag{5.5.2}$$

Ja akolouj soume touc upologismoÔc sto ˆ rjro [1].

AnaptÔssontac touc ekjetikoÔc par ˆ gontec stic katast ˆ seic twn nef¸n, paÐrnoume

$$\tilde{S}_{\alpha\beta} = N_\alpha N_\beta \sum_{n,m=0}^{\infty}\frac{1}{m!n!}\langle\beta|\prod_{l=1}^{m}\int_\lambda^{E_d}\frac{d^3\vec{q}_l}{(2\pi)^3}\frac{f_\beta^*(\vec{q}_l)\cdot a(\vec{q}_l)}{(2\omega_{\vec{q}_l})} S \prod_{s=1}^{m}\int_\lambda^{E_d}\frac{d^3\vec{k}_s}{(2\pi)^3}\frac{f_\beta(\vec{k}_s)\cdot a^\dagger(\vec{k}_s)}{(2\omega_{\vec{k}_s})}|\alpha\rangle \tag{5.5.3}$$

K ˆ je ìroc se aut thn èkfrash apoteleÐ pl ˆ toc skèdashc me n eiserqìmena kai m exerqìmena upèrujra fwtìnia stajmismèno me ton par ˆ gonta $1/m!n!$. E ˆ n l apì aut ˆ ta fwtìnia den allhlepidroÔn me ta fortismèna swmatÐdia, tìte $n' = n - l$ upèrujra fwtìnia aporrofoÔntai apì mia exwterik gramm kai $m^{'} = m - l$ upèrujra fwtìnia ekpèmpontai apì mia exwterik gramm .

Ta l mh allhlepidr¸nta fwtìnia suneisfèroun me èna par^gonta

$$l!\left(\int_{\lambda}^{E_d} \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} f_{\alpha}^{\mu}(\vec{q}) f_{\beta\mu}^{*}(\vec{q})\right) \tag{5.5.4}$$

Ajroðzoume wc proc tic egk^rsiec pol¸seic twn fwtonðwn ($r = 1, 2$)

$$\sum_{r} \epsilon_{r\mu}(\vec{q})\epsilon_{r\nu}^{*}(\vec{q}) = \eta_{\mu\nu} - q_{\mu}c_{\nu} - q_{\nu}c_{\mu} \tag{5.5.5}$$

kai qrhsimopoioÔme thn idiìthta ìti oi sunart seic ntusðmatoc eðnai egk^rsiec ($f_{\beta}^{*}q = f_{\alpha}q = 0$).

Jewr¸ntac thn enèrgeia E_d aujaðreta mikr , mporoÔme na upologðsoume tic suneisforèc twn allhlepidr¸ntwn upèrujrwn fwtonðwn qrhsimopoi¸ntac ta akìlouja upèrujra jewr mata [13, 14, 15]

$$\lim_{|\vec{q}|\to 0} (2\omega_{\vec{q}})^{\frac{1}{2}} \langle\beta|a_r(\vec{q})S|\alpha\rangle = \left(\sum_{i\in\beta}\frac{e_i p_i \epsilon_r^{*}(\vec{q})}{p_i q} - \sum_{i\in\alpha}\frac{e_i p_i \epsilon_r^{*}(\vec{q})}{p_i q}\right)\langle\beta|S|\alpha\rangle \tag{5.5.6}$$

kai

$$\lim_{|\vec{q}|\to 0} (2\omega_{\vec{q}})^{\frac{1}{2}} \langle\beta|Sa_r^{\dagger}(\vec{q})|\alpha\rangle = -\left(\sum_{i\in\beta}\frac{e_i p_i \epsilon_r(\vec{q})}{p_i q} - \sum_{i\in\alpha}\frac{e_i p_i \epsilon_r(\vec{q})}{p_i q}\right)\langle\beta|S|\alpha\rangle \tag{5.5.7}$$

To pl^toc $\tilde{S}_{\alpha\beta}$ mporeð na upologisteð ajroðzontac wc proc ìlec tic epitreptèc di-at^xeic $m^{'}, n^{'}, l$, lamb^nontac upìyin ìlouc touc pollaplasiastikoÔc par^gontec b^rouc. Up^rqoun $(n^{'} + l)!/(n^{'})!(l)!$ trìpoi gia na epilegoÔn l fwtìnia apì ta n fwtìnia thc ar-qik c kat^stashc kai antðstoiqa $(m^{'} + l)!/(m^{'})!(l)!$ trìpoi gia ta fwtìnia thc telik c kat^stashc. Epomènwc

$$\begin{aligned}
\tilde{S}_{\alpha\beta} = N_{\alpha}N_{\beta} &\sum_{n^{'},m^{'},l=0}^{\infty} \frac{1}{(m^{'}+l)!(n^{'}+l)!}\frac{(n^{'}+l)!}{n^{'}!l!}\frac{(m^{'}+l)!}{m^{'}!l!}l! \\
\times \left(\int_{\lambda}^{E_d} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} f_{\alpha}^{\mu}(\vec{q}) f_{\beta\mu}^{*}(\vec{q})\right) &\left[\int_{\lambda}^{E_d} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} \sum_{i\in\{\alpha,\beta\}} e_i\eta_i \left(\frac{f_{\beta}^{*}(\vec{q})p_i}{(p_i q)} - f_{\beta}^{*}(\vec{q})c\right)\right]^{m^{'}} \\
&\times \left[-\int_{\lambda}^{E_d} \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}} \sum_{i\in\{\alpha,\beta\}} e_i\eta_i \left(\frac{f_{\alpha}(\vec{k})p_i}{(p_i k)} - f_{\alpha}(\vec{k})c\right)\right]^{n^{'}} S_{\alpha\beta}
\end{aligned} \tag{5.5.8}$$

'Oroi an^logoi me to di^nusma c mhdenÐzontai lìgw diat rhshc fortÐou. Oi seirèc ekjetopoioÔntai kai epomènwc katal goume sthn èkfrash

$$\tilde{S}_{\alpha\beta} = N_\alpha N_\beta e^{\int_\lambda^{E_d} \frac{d^3\vec{q}}{(2\pi)^3}\frac{1}{2\omega_{\vec{q}}} f^\mu_\alpha(\vec{q}) f^*_{\beta\mu}(\vec{q})} e^{\int_\lambda^{E_d} \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}} \sum_{ij} e_i e_j \eta_i \eta_j \frac{p_i p_j}{(p_i q)(p_j q)}} S_{\alpha\beta} \tag{5.5.9}$$

Oi pr¸toi treic ìroi dÐnoun to pl^toc allhlepik^luyhc $\langle f_\beta | f_\alpha \rangle$, en¸ to deÔtero ekjetikì dÐnei

$$\int_\lambda^{E_d} \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega_{\vec{k}}} \sum_{ij} e_i e_j \eta_i \eta_j \frac{p_i p_j}{(p_i q)(p_j q)} = \log\left(\frac{E_d}{\lambda}\right) 2\mathcal{B}_{\alpha\beta} \tag{5.5.10}$$

ìpou o suntelest c $\mathcal{B}_{\alpha\beta}$ dÐnetai apì thn èkfrash 5.4.13. Sunep¸c

$$\tilde{S}_{\alpha\beta} = \langle f_\beta | f_\alpha \rangle \left(\frac{E_d}{\lambda}\right)^{2\mathcal{B}_{\alpha\beta}} S_{\alpha\beta} = \left(\frac{E_d}{\lambda}\right)^{\mathcal{B}_{\alpha\beta}} S_{\alpha\beta} \tag{5.5.11}$$

'Opwc anafèrame prohgoumènwc, oi logarijmikoÐ apeirismoÐ exaitÐac twn eikonik¸n fwtonÐwn ekjetopoioÔntai, me apotèlesma [12]

$$S_{\alpha\beta} = \left(\frac{\lambda}{\Lambda}\right)^{\mathcal{B}_{\alpha\beta}} e^{i\varphi_{\alpha\beta}} S^{(\Lambda)}_{\alpha\beta} \tag{5.5.12}$$

ìpou to pinakostoiqeÐo $S^{(\Lambda)}_{\alpha\beta}$ den perilamb^nei tic suneisforèc eikonik¸n fwtonÐwn me orm mikrìterh apì Λ. H $\varphi_{\alpha\beta}$ eÐnai mia pragmatik f^sh [12] kai den suneisfèrei sthn apìluth tim tou pl^touc. Epomènwc h posìthta

$$\tilde{S}_{\alpha\beta} = \left(\frac{E_d}{\Lambda}\right)^{\mathcal{B}_{\alpha\beta}} e^{i\varphi_{\alpha\beta}} S^{(\Lambda)}_{\alpha\beta} \tag{5.5.13}$$

eÐnai peperasmènh kai genik^ mh mhdenik . Sto ìrio $\lambda \to 0$, krat¸ntac to phlÐko E_d/Λ stajerì.

5.5.1 Paragwg enìc upèrujrou fwtonÐou

Se autì to kef^laio ja melet soume thn perÐptwsh ìpou èqoume èna epiprìsjeto ekpempìmeno upèrujro fwtìnio sthn telik kat^stash, me orm $\vec{q}_\gamma$ kai di^nusma pìlwshc $\epsilon^\mu_r(\vec{q}_\gamma)$, ¸ste $|\vec{q}_\gamma| < E_d$:

$$\tilde{S}_{\beta\gamma,\alpha} =_d \langle \beta\gamma | S | \alpha \rangle_d \tag{5.5.14}$$

H perÐptwsh $|\vec{q}_\gamma| > E_d$ kalÔptetai apì thn prohgoÔmenh an^lush.

Arqik^ shmei¸noume ìti

$$|\beta\gamma\rangle_d = e^{R_f}|\beta\gamma\rangle = e^{R_f} a_r^\dagger(\vec{q}_\gamma)|\beta\rangle = \Big(|\beta\gamma\rangle - f_\beta^{*\mu}(\vec{q}_\gamma)\epsilon_{r\mu}(\vec{q}_\gamma)|\beta\rangle\Big) \times |f_\beta\rangle \tag{5.5.15}$$

O deÔteroc ìroc sthn parènjesh proèrqetai apì thn met^jesh tou telest R_f me ton telest dhmiourgÐac tou epiprìsjetou fwtonÐou. O telest c R_f perièqei telestèc dhmiourgÐac kai katastrof c fwtonÐwn me enèrgeia mikrìterh apì E_d. To tetrimmèno mèroc tou pÐnaka skèdashc suneisfèrei me to ginìmeno $_d\langle\beta\gamma|\alpha\rangle_d$, to opoÐo mhdenÐzetai

$$_d\langle\beta\gamma|\alpha\rangle_d = \Big(f_\alpha^\mu(\vec{q}_\gamma) - f_\beta^\mu(\vec{q}_\gamma)\Big)\,\epsilon_{\mu r}^*(\vec{q}_\gamma)\langle f_\beta|f_\alpha\rangle\langle\beta|\alpha\rangle = \left(f_\alpha^\mu(\vec{q}_\gamma) - f_\alpha^\mu(\vec{q}_\gamma)\right)\epsilon_{\mu r}^*(\vec{q}_\gamma) = 0 \tag{5.5.16}$$

ìpou qrhsimopoi same thn sqèsh orjokanonikìthtac $\langle\beta|\alpha\rangle = \delta_{\alpha\beta}$. 'Etsi mìno to mh tetrimmèno komm^ti tou pÐnaka skèdashc suneisfèrei sto pio p^nw pinakostoiqeÐo.

MporoÔme na gr^youme to pl^toc skèdashc $\tilde{S}_{\beta\gamma,\alpha}$ wc ^jroisma dÔo ìrwn

$$\tilde{S}_{\beta\gamma,\alpha} = \tilde{S}^{(1)}_{\beta\gamma,\alpha} + \tilde{S}^{(2)}_{\beta\gamma,\alpha} \tag{5.5.17}$$

ìpou

$$\tilde{S}^{(1)}_{\beta\gamma,\alpha} = -f_\beta(\vec{q}_\gamma)\cdot\epsilon_r^*(\vec{q}_\gamma)\tilde{S}_{\beta\alpha} = -f_\beta(\vec{q}_\gamma)\cdot\epsilon_r^*(\vec{q}_\gamma)\left(\frac{E_d}{\lambda}\right)^{\mathcal{B}_{\alpha\beta}} S_{\beta\alpha} \tag{5.5.18}$$

kai

$$\tilde{S}^{(2)}_{\alpha\beta} = N_\alpha N_\beta \sum_{n,m=0}^{\infty} \frac{(2\omega_\gamma)^{\frac{1}{2}}}{m!n!}$$
$$\times\langle\beta|a_r(\vec{q}_\gamma)\prod_{l=1}^{m}\int_\lambda^{E_d}\frac{d^3\vec{q}_l}{(2\pi)^3}\frac{f_\beta^*(\vec{q}_l)\cdot a(\vec{q}_l)}{(2\omega_{\vec{q}_l})^{1/2}} S \prod_{s=1}^{m}\int_\lambda^{E_d}\frac{d^3\vec{k}_s}{(2\pi)^3}\frac{f_\beta(\vec{k}_s)\cdot a^\dagger(\vec{k}_s)}{(2\omega_{\vec{k}_s})^{1/2}}|\alpha\rangle \tag{5.5.19}$$

Akolouj¸ntac parìmoia diadikasÐa me thn perÐptwsh tou upologismoÔ tou $\tilde{S}_{\alpha\beta}$ (exÐswsh 5.5.1) katal goume sthn sqèsh [1]

$$\tilde{S}^{(2)}_{\alpha\beta} = \left(\frac{E_d}{\lambda}\right)^{\mathcal{B}_{\alpha\beta}} S_{\beta\alpha} f_\beta(\vec{q}_\gamma)\cdot\epsilon_r^*(\vec{q}_\gamma) + \ldots \tag{5.5.20}$$

ìpou oi teleÐec sumperilamb^noun ìrouc pou den parousi^zoun apoklÐseic.

AjroÐzontac tic sqèseic 5.5.18 kai 5.5.20 parathroÔme ìti ìloi oi ìroi pou apoklÐnoun anairoÔntai kai mporoÔme na gr^youme

$$\tilde{S}_{\beta\gamma,\,\alpha} = F_{\beta\alpha}(\vec{q}_\gamma, \epsilon_r(\vec{q}_\gamma)) \tag{5.5.21}$$

ìpou $F_{\beta\alpha}(\vec{q}_\gamma, \epsilon_r(\vec{q}_\gamma))$ mia omal sun^rthsh sto ìrio $\lambda, |\vec{q}_\lambda| \to 0$. MporeÐ na deiqteÐ ìti aut h sun^rthsh eÐnai thc t^xhc E_d [26]. Epomènwc to ntÔsimo katastèllei thn ekpomp upèrujrwn fwtonÐwn me enèrgeia $\omega_\gamma < E_d$, toul^qiston se epÐpedo diagr^mmatoc dèntrou.

5.6 DiakritopoÐhsh

Gia ton upologismì thc entropÐac sÔmplexhc sumpagopoioÔme ton ^peiro q¸ro se èna meg^lo koutÐ m kouc L ($V = L^3$) kai epib^lloume periodikèc sunoriakèc sunj kec. H orm twn swmatidÐwn eÐnai kbantwmènh

$$\vec{k} = \frac{2\pi}{L}(n_1, n_2, n_3) \tag{5.6.1}$$

Oi telestèc dhmiourgÐac-katastrof c sto suneqèc sundèontai me autoÔc sto koutÐ wc ex c

$$a_r(\vec{k}) \to V^{\frac{1}{2}} \tilde{a}_r(\vec{k}) \tag{5.6.2}$$

ètsi h sqèsh met^jeshc an^getai sthn akìlloujh

$$[\tilde{a}_r(\vec{k}), \tilde{a}_s^\dagger(\vec{q})] = \delta_{rs}\, \delta_{\vec{k}\vec{q}} \tag{5.6.3}$$

H kat^stash enìc fwtonÐou

$$\tilde{a}_s^\dagger(\vec{k})|0\rangle \tag{5.6.4}$$

eÐnai kanonikopoihmènh sthn mon^da. To upèrujro ìrio apokop c λ sundèetai me to m koc tou koutioÔ mèsw thc sqèshc $\lambda = 2\pi/L$. Gia q^rin aplìthtac ja agno soume to sÔmbolo thc perispwmènhc stouc telestèc dhmiourgÐac-katastrof c sto koutÐ.

Mia arqik kat^stash $|\beta\rangle$ pou apoteleÐtai apì uperi¸dh swmatÐdia enèrgeiac $E_\beta > E$ ntÔnetai me ton akìloujo trìpo

$$|\beta\rangle_d = |\beta\rangle_H \times |f_\beta\rangle_S \tag{5.6.5}$$

Sthn perÐptwsh pou o q¸roc twn orm¸n eÐnai diakritopoihmènoc, h kat^stash tou nèfouc twn fwtonÐwn $|f_\beta\rangle_S$ dÐnetai apì thn èkfrash

$$|f_\beta\rangle_S = U_\beta|0\rangle_S = N_\beta e^{A_\beta^\dagger}|0\rangle_S \tag{5.6.6}$$

ìpou

$$U_\beta = e^{A_\beta^\dagger - A_\beta}, \quad A_\beta = \sum_{\omega_{\vec{k}} < E_d} \frac{1}{(2V\omega_{\vec{k}})^{1/2}} f_\beta^*(\vec{k}) \cdot a(\vec{k}) \tag{5.6.7}$$

$$N_\beta = e^{-\frac{1}{2}\sum_{\omega_{\vec{k}}<E_d}\frac{1}{2V\omega_{\vec{k}}}f^\mu_\beta(\vec{k})f^*_{\beta\mu}(\vec{k})} \tag{5.6.8}$$

H kat^stash $|f_\beta\rangle$ an kei sto upèrujro mèroc tou q¸rou Hilbert.

5.7 'Iqnoc wc proc upèrujrouc bajmoÔc eleujerÐac

Ac jewr soume ton telest ket-bra

$$|\beta\rangle_d\langle\beta^{'}|_d \tag{5.7.1}$$

PaÐrnontac to merikì Ðqnoc wc proc to upèrujro mèroc tou q¸rou Hilbert paÐrnoume

$$\mathrm{Tr}_{H_S}(|\beta\rangle_d\langle\beta^{'}|_d) = |\beta\rangle_H\langle\beta^{'}|_H\langle f_{\beta'}|f_\beta\rangle \tag{5.7.2}$$

H pl^toc epik^luyhc $\langle f_{\beta'}|f_\beta\rangle$ sto suneqèc ìrio dÐnetai apì thn èkfrash 5.4.13. E^n $\beta\neq\beta^{'}$, sto ìrio $\lambda\to 0$ kai se ìlec tic t^xeic, to pl^toc epik^luyhc mhdenÐzetai. Wc apotèlesma gia opoiad pote upèr jesh ntumènwn katast^sewn, to merikì Ðqnoc wc proc to upèrujro mèroc tou q¸rou Hilbert odhgeÐ se aposunoq kai èna (sqedìn) diag¸nio pÐnaka puknìthtac [27].

'Estw t¸ra ìti pros jètoume èna upèrujro fwtìnio sthn kat^stash $|\beta\rangle$ me orm $\vec{q}$, $|\vec{q}|<E$ kai di^nusma pìlwshc $\epsilon^\mu_r(\vec{q})$:

$$|\beta\gamma(\vec{q},r)\rangle = a^\dagger_r(\vec{q})|\beta\rangle \tag{5.7.3}$$

H antÐstoiqh ntumènh kat^stash gr^fetai

$$|\beta\gamma(\vec{q},r)\rangle_d = |\beta\rangle_H\times(U_\beta a^\dagger_r(\vec{q})|0\rangle_S) = |\beta\rangle_H\times(a^\dagger_r(\vec{q}) - [A_\beta, a^\dagger_r(\vec{q})])|f_\beta\rangle_S \tag{5.7.4}$$

Me b^sh aut thn èkfrash upologÐzoume to merikì Ðqnoc

$$\mathrm{Tr}_{H_S}\left(|\beta\gamma\rangle_d\langle\beta^{'}|_d\right) = |\beta\rangle_H\langle\beta^{'}|_H\langle f_{\beta'}|f_\beta\rangle\left([A_{\beta'}, a^\dagger_r(\vec{q})] - [A_\beta, a^\dagger_r(\vec{q})]\right) \tag{5.7.5}$$

E^n $E_d<|\vec{q}|<E$, oi metajètec mhdenÐzontai epeid oi telestèc katabÐbashc tou A_β èqoun ormèc $|\vec{q}|<E_d$. E^n $|\vec{q}|<E_d$, paÐrnoume

$$\mathrm{Tr}_{H_S}\left(|\beta\gamma\rangle_d\langle\beta^{'}|_d)\right) = |\beta\rangle_H\langle\beta^{'}|_H\langle f_{\beta'}|f_\beta\rangle\frac{1}{(2V\omega_{\vec{q}})^{1/2}}\left(f^*_{\beta'}(\vec{q}) - f^*_\beta(\vec{q})\right)\cdot\epsilon_r(\vec{q}) \tag{5.7.6}$$

E^n $\beta=\beta^{'}$ h èkfrash mhdenÐzetai. H sun^rthsh $f_\beta(\vec{q})$ eÐnai an^logh tou fortÐou e.

Tèloc ja qreiastoÔme thn sqèsh

$$\mathrm{Tr}_{H_S}\left(|\beta\gamma\rangle_d\langle\beta^{'}\gamma^{'}|_d\right) = |\beta\rangle_H\langle\beta^{'}|_H\langle f_{\beta'}|f_\beta\rangle$$
$$\times\left\{\delta_{rs}\delta_{\vec{q}\vec{k}} + \left([A^\dagger_{\beta'}, a_s(\vec{k})] - [A^\dagger_\beta, a_s(\vec{k})]\right)\left([A_{\beta'}, a^\dagger_r(\vec{q})] - [A_\beta, a^\dagger_r(\vec{q})]\right)\right\} \tag{5.7.7}$$

ìpou to fwtìnio γ' èqei orm $\vec{k}$ kai elikìthta s. E^n $|\vec{q}|, |\vec{k}| < E_d$, paÐrnoume

$$\mathrm{Tr}_{H_S}\left(|\beta\gamma\rangle_d\langle\beta^{'}\gamma^{'}|_d\right) = |\beta\rangle_H\langle\beta^{'}|_H\langle f_{\beta'}|f_\beta\rangle$$
$$\times\left\{\delta_{rs}\delta_{\vec{q}\vec{k}} + \frac{1}{(2V\omega_{\vec{q}})^{1/2}2V\omega_{\vec{k}})^{1/2}}\left(f^*_\beta(\vec{k}) - f^*_{\beta'}(\vec{k})\right)\cdot\epsilon_s(\vec{k})\left(f^*_{\beta'}(\vec{q}) - f^*_\beta(\vec{q})\right)\cdot\epsilon_r(\vec{q})\right\} \tag{5.7.8}$$

ParomoÐwc mporoÔme na upologÐsoume merik^ Ðqnh kai se peript¸seic pou dÔo peris-sìtera upèrujra fwtìnia emfanÐzontai sthn arqik^ gumn kat^stash. Ja emfanÐzontai ìroi me megalÔterh t^xh wc proc $f_\beta(\vec{q})$.

Kef^laio 6

Skèdash me ntumènec katast^seic kai entropÐa sÔmplexhc

'Estw ìti èqoume mia arqik kat^stash $|\alpha\rangle$ me aujaÐreto arijmì fortismènwn swmatidÐwn (hlektronÐwn kai pozitronÐwn sthn perÐptwsh thc QED).

$$|\Psi\rangle_{in} = |\alpha\rangle_d = |\alpha\rangle_H \times |f_\alpha\rangle_S \tag{6.1}$$

Aut eÐnai mia kajar kat^stash kai den up^rqei sÔmplexh metaxÔ twn uperÔjrwn kai twn uperiwd¸n bajm¸n eleujerÐac. O pÐnakac puknìthtac twn uperiwd¸n bajm¸n eleujerÐac mporeÐ na grafteÐ sthn morf

$$\rho_H = \mathrm{Tr}_{H_S}(|\Psi\rangle_{in}\langle\Psi|_{in}) = |\alpha\rangle_H\langle\alpha|_H \tag{6.2}$$

kai perigr^fei mia kajar kat^stash. SÔmplexh emfanÐzetai mèsw thc diadikasÐac thc skèdashc.

H telik kat^stash sundèetai me thn arqik kat^stash mèsw thc dr^shc pou pÐnaka skèdashc S

$$|\Psi\rangle_{out} = S|\Psi\rangle_{in} = (1 + iT)|\alpha\rangle_d \tag{6.3}$$

Epeid o pÐnakac S eÐnai monadiakìc ($S^\dagger S = 1$), isqÔei h akìloujh sqèsh (sqèsh monadiakìthtac):

$$i(T - T^\dagger) + T^\dagger T = 0 \tag{6.4}$$

Eis^gontac mia pl rh b^sh ntumènwn katast^sewn mporoÔme na gr^youn thn telik kat^stash $|\Psi\rangle_{out}$ wc akoloÔjwc

$$|\Psi\rangle_{out} = |\alpha\rangle_d + \tilde{A}_{\beta\alpha}|\beta\rangle_d + \tilde{B}_{\beta\gamma,\alpha}|\beta\gamma\rangle_d + \dots \tag{6.5}$$

ìpou $\tilde{A}_{\beta\alpha} =_d \langle\beta|\, iT|\alpha\rangle_d$ kai $\tilde{B}_{\beta\gamma,\alpha} =_d \langle\beta\gamma|iT|\alpha\rangle_d$ eÐnai pl^th skèdashc metaxÔ twn ntumènwn katast^sewn[1]. EnnoeÐtai ^jroisma wc proc tic epitreptèc telikèc katast^-seic tou sust matoc β, tìso wc proc touc diaforetikoÔc sunduasmoÔc swmatidÐwn pou mporeÐ na èqei h telik kat^stash ìso kai wc proc tic ormèc, ta spin kai thn elikìthta twn telik¸n swmatidÐwn. EpÐshc ^jroisma uponoeÐtai kai wc proc tic katast^seic twn ekpempìmenwn fwtonÐwn γ.

Oi katast^seic $|\beta\rangle$ perilamb^noun ìlec tic epitrepìmenec telikèc katast^seic qwrÐc fwtìnia. Perilamb^noun epÐshc telikèc katast^seic me ton mikrìtero arijmì fwtonÐwn pou eÐnai dunatìn na prokÔyoun met^ apì èna sugkekrimèno arijmì exaôl¸sewn hlek-tronÐwn/pozitronÐwn. Ta fwtìnia aut^ eÐnai uperi¸dh. Gia par^deigma ac jewr soume mia arqik kat^stash me èna hlektrìnio kai èna pozitrìnio. Oi katast^seic $|\beta\rangle$ peril-amb^noun ìlec tic katast^seic hlektronÐou/pozitronÐou pou prokÔptoun wc apotèlesma skèdashc, kaj¸c kai katast^seic dÔo fwtonÐwn pou prokÔptoun wc apotèlesma exaÔl-wshc. Oi katast^seic $|\beta\gamma\rangle$ perilamb^noun èna epiprìsjeto fwtìnio to opoÐo ekpèmpetai apì èna fortismèno swmatÐdio. (Perissìtera paradeÐgmata parousi^zontai sto epìmeno kefalaÐou.)

Ta "^ntuta" pl^th skèdashc $A_{\beta\alpha}$ kai $B_{\beta\gamma,\alpha}$ dÐdontai sunart sei sundedemènwn kai asÔndetwn diagramm^twn Feynman. H kÔria suneisfor^ sto pl^toc $A_{\beta\alpha}$ proèrqetai apì èna asÔndeto di^gramma t^xhc e^2 me ìla ta ^lla swmatÐdia na eÐnai asÔndeta. Pio genik^ ja sumbolÐsoume thn t^xh tou pl^touc $A_{\beta\alpha}$ me e^N, ìpou o arijmìc N exart^tai apì thn kat^stash $|\beta\rangle$. To pl^toc $|\beta\gamma\rangle$ eÐnai t^xhc e^{N+1}

Gia par^deigma, e^n h arqik kat^stash apoteleÐ èna zeÔgoc hlektronÐou/pozitronÐou, tìte gia tic diergasÐec $e^-e^+ \to e^-e^+$ kai $e^-e^+ \to \gamma\gamma$ ta kÔria diagr^mmata Feynman pou suneisfèroun eÐnai thc t^xhc e^2 (èqoun dÔo korufèc antÐstoiqa), en¸ gia thn diergasÐa $e^-e^+ \to e^-e^+\gamma$ to kÔrio di^gramma eÐnai thc t^xhc e^3 afoÔ èqei 3 korufèc.

O pÐnakac puknìthtac pou antistoiqeÐ sthn telik kat^stash dÐdetai apì thn akìlouj h èkfrash

$$\begin{aligned}|\Psi\rangle_{out}\langle\Psi|_{out} &= |\alpha\rangle_d\langle\alpha|_d + \left(\tilde{A}_{\beta\alpha}|\beta\rangle_d + \tilde{B}_{\beta\gamma,\alpha}|\beta\gamma\rangle_d + \dots\right)\langle\alpha|_d \\ &+ |\alpha\rangle_d\left(\tilde{A}^*_{\beta'\alpha}\langle\beta'|_d + \tilde{B}^*_{\beta'\gamma',\alpha}\langle\beta'\gamma'|_d + \dots\right) + \tilde{A}_{\beta\alpha}\tilde{A}^*_{\beta'\alpha}|\beta\rangle_d\langle\beta'|_d \\ &+ \tilde{B}_{\beta\gamma,\alpha}\tilde{B}^*_{\beta'\gamma',\alpha}|\beta\gamma\rangle_d\langle\beta'\gamma'|_d + \tilde{B}_{\beta\gamma,\alpha}\tilde{A}^*_{\beta'\alpha}|\beta\gamma\rangle_d\langle\beta'|_d + \tilde{A}_{\beta\alpha}\tilde{B}^*_{\beta'\gamma',\alpha}|\beta\rangle_d\langle\beta'\gamma'|_d + \dots\end{aligned} \tag{6.6}$$

[1] Οι τελείες περιλαμβάνουν όρους μεγαλύτερης τάξης ως προς e, οι οποίοι όπως θα δούμε δεν συνεισ-φέρουν στην κύρια τάξη της εντροπίας σύμπλεξης.

6.1 PÐnakac puknìthtac twn uperiwd¸n bajm¸n eleujerÐac

Sth sunèqeia paÐrnoume to merikì Ðqnoc wc proc ìlouc touc upèrujrouc bajmoÔc eleujerÐac. Se autoÔc perilamb^nontai ta fwtìnia sta nèfh, kaj¸c kai ekpempìmena fwtìnia me enèrgeia mikrìterh apì enèrgeia E. O pÐnakac puknìthtac dÐnetai apì thn èkfrash

$$\rho_H = \mathrm{Tr}_{H_S}\left(|\Psi\rangle_{out}\langle\Psi|_{out}\right) \tag{6.1.1}$$

Qrhsimopoi¸ntac tic ekfr^seic 5.7.2, 5.7.6 kai 5.7.8 paÐrnoume

$$\begin{aligned}\rho_H = {} & |\alpha\rangle_H\langle\alpha|_H + \left(C_\beta|\beta\rangle_H + \sum_{\omega_\gamma>E} C_{\beta\gamma}|\beta\gamma\rangle_H + \dots\right)\langle\alpha|_H \\ & +|\alpha\rangle_H\left(C^*_{\beta'}\langle\beta'|_H + \sum_{\omega_{\gamma'}>E} C^*_{\beta'\gamma'}\langle\beta'\gamma'|_H + \dots\right) \\ & +D_{\beta,\beta'}|\beta\rangle_H\langle\beta'|_H + \sum_{\omega_{\gamma'},\omega_\gamma>E} D_{\beta\gamma,\beta'\gamma'}|\beta\gamma\rangle_H\langle\beta'\gamma'|_H \\ & + \sum_{\omega_\gamma>E} D_{\beta\gamma,\beta'}|\beta\gamma\rangle_H\langle\beta'|_H + \sum_{\omega_{\gamma'}>E} D_{\beta'\gamma',\beta}|\beta\rangle_H\langle\beta'\gamma'|_H + \dots\end{aligned} \tag{6.1.2}$$

ìpou

$$\frac{C_\beta}{\langle f_\alpha|f_\beta\rangle} = \tilde{A}_{\beta\alpha} + \sum_{\omega_\gamma<E_d}\frac{1}{(2V\omega_\gamma)^{\frac{1}{2}}}\tilde{B}_{\beta\gamma,\alpha}\left(f^*_\alpha(\vec{q}_\gamma) - f^*_\beta(\vec{q}_\gamma)\right)\cdot\epsilon(\gamma) + \dots \tag{6.1.3}$$

$$C_{\beta\gamma} = \langle f_\alpha|f_\beta\rangle\tilde{B}_{\beta\gamma,\alpha} + \dots \tag{6.1.4}$$

$$\begin{aligned}\frac{D_{\beta,\beta'}}{\langle f_{\beta'}|f_\beta\rangle} = {} & \tilde{A}_{\beta\alpha}\tilde{A}^*_{\beta'\alpha} + \sum_{\omega_\gamma<E_d}\frac{1}{(2V\omega_\gamma)^{\frac{1}{2}}}\tilde{B}_{\beta\gamma,\alpha}\tilde{A}^*_{\beta'\alpha}\left(f^*_{\beta'}(\vec{q}_\gamma) - f^*_\beta(\vec{q}_\gamma)\right)\cdot\epsilon(\gamma) \\ & + \sum_{\omega'_\gamma<E_d}\frac{1}{(2V\omega_\gamma)^{\frac{1}{2}}}\tilde{B}^*_{\beta'\gamma',\alpha}\tilde{A}_{\beta\alpha}\left(f_\beta(\vec{q}_{\gamma'}) - f_{\beta'}(\vec{q}_{\gamma'})\right)\cdot\epsilon^*(\gamma') + \sum_{\omega_\gamma<E}\tilde{B}_{\beta\gamma,\alpha}\tilde{B}^*_{\beta'\gamma,\alpha} + \dots\end{aligned} \tag{6.1.5}$$

$$D_{\beta\gamma,\beta'} = \langle f_{\beta'}|f_\beta\rangle\tilde{B}_{\beta\gamma,\alpha}\tilde{A}^*_{\beta'\alpha} + \dots \tag{6.1.6}$$

$$D_{\beta\gamma,\beta'\gamma'} = \langle f_{\beta'}|f_\beta\rangle\tilde{B}_{\beta\gamma,\alpha}\tilde{B}^*_{\beta'\gamma',\alpha}\dots \tag{6.1.7}$$

Ta stoiqeĐa tou pĐnaka ρ_H dĐdontai sunart sei ntumènwn plat¸n skèdashc, ta opoĐa den perièqoun logarijmikèc apoklĐseic wc proc λ, kaj¸c kai pl^th epik^luyhc pou sundèontai me ta nèfh. Ta diag¸nia stoiqeĐa eĐnai an^loga me tic (inclusive) pijanìthtec (an^ mon^da qrìnou) Bloch - Nordsieck, kai den parousi^zoun upèrujrec apoklĐseic, se ìlec tic t^xeic sthn jewrĐa diataraq¸n [1].

Ta mh diag¸nia stoiqeĐa tou pĐnaka puknìthtac eĐnai an^loga twn plat¸n epik^luyhc $\langle f_\beta | f_{\beta'} \rangle$, ta opoĐa, se opoiad pote peperasmènh t^xh sth jewrĐa diataraq¸n, perièqoun logarijmikèc apoklĐseic wc proc λ. E^n oi ormèc twn swmatidĐwn β kai β' diafèroun, tìte to pl^toc epik^luyhc mhdenĐzetai se ìlec tic t^xeic, sto ìrio $\lambda \to 0$. Se autì to ìrio, o pĐnakac puknìthtac eĐnai sqedìn diag¸nioc kai ekdhl¸nei aposunoq [27]. ExakoloujeĐ na up^rqei mh tetrimmènh ex^rthsh apì ton ìgko sta mh mhdenik^ stoiqeĐa.

Stouc akìloujouc upologismoÔc krat^me ton ìgko tou koutioÔ, kai epomènwc to ìrio apokop c λ, peperasmèno kai upologĐzoume tic di^forec posìthtec diataraktik^, anaptÔssontac se sugkekrimènh t^xh sth jewrĐa diataraq¸n. Epomènwc prèpei na sunupologĐsoume th suneisfor^ kai twn mh diag¸niwn stoiqeĐwn. PaĐrnoume to suneqèc ìrio $\lambda \to 0$ sto tèloc.

H kÔria t^xh tou ìrou C_β eĐnai e^N, tou ìrou $C_{\beta\gamma}$ eĐnai e^{N+1}, tou $D_{\beta\beta'}$ eĐnai e^{2N}, tou $D_{\beta\gamma,\beta'}$ eĐnai e^{2N+1} kai tou $D_{\beta\gamma,\beta'\gamma'}$ eĐnai e^{2N+2}. Oi teleĐec stic ekfr^seic 6.1.2-6.1.6 perilamb^noun ìrouc pou den suneisfèroun sthn kÔria t^xh thc entropĐac sÔmplexhc. EpĐshc sthn èkfrash tou suntelest C_β, o deÔteroc ìroc mhdenĐzetai e^n $\beta = \alpha$. Oi upìloipoi ìroi eĐnai an^logoi me ginìmena tou pl^touc $B_{\beta\gamma_1\ldots\gamma_i\ldots}$ kai ìrwn thc morf c $(f_\alpha(\vec{q}_i) - f_\beta(\vec{q}_i))$, oi opoĐoi mhdenĐzontai epĐshc gia $\beta = \alpha$. Epomènwc èqoume $C_\alpha = \tilde{A}_{\alpha\alpha}$ se ìlec tic t^xeic. Tèloc o ìroc $C_\alpha + C^*_\alpha$ eĐnai t^xhc e^4 exaitĐac thc sqèshc monadiakìthtac ($S^\dagger S = 1$).

6.2 Upologismìc thc t^xhc thc entropĐac sÔmplexhc

Sto kef^laio autì prosdiorĐzoume thn kÔria t^xh thc entropĐac sÔmplexhc ¸ste na amel soume ìrouc an¸terhc t^xhc. Gia ton skopì autì gr^youme ton pĐnaka puknìthtac sthn morf

$$\rho_H = q|\varphi\rangle\langle\varphi| + \mathcal{O}(e^n) \tag{6.2.1}$$

ìpou $|\varphi\rangle$ eĐnai mia kanonikopoihmènh kat^stash. Tìte o pĐnakac puknìthtac mporeĐ na grafteĐ sthn morf

$$\rho_H = \begin{pmatrix} q & \cdots \\ \vdots & \ddots \end{pmatrix} \tag{6.2.2}$$

ìpou oi teleÐec perilamb^noun ìrouc t^xhc $\mathcal{O}(e^n)$.

Epeid to Ðqnoc tou pÐnaka puknìthtac isoÔtai me thn mon^da o suntelest c q eÐnai thc t^xhc $q = 1 - \mathcal{O}(e^n)$. Autìc o pÐnakac èqei mia idiotim t^xhc $1 - \mathcal{O}(e^n)$ kai oi upìloipec idiotimèc tou eÐnai thc t^xhc e^n. Epomènwc h entropÐa sÔmplexhc paÐrnei thn morf

$$S_{ent} = -\sum_i \lambda_i \log \lambda_i = \left(1 - \mathcal{O}(e^n)\right) \log \left(1 - \mathcal{O}(e^n)\right) + \mathcal{O}(e^n) \log \mathcal{O}(e^n) \tag{6.2.3}$$

Qrhsimopoi¸ntac to an^ptugma gia ton log^rijmo $\log(1-x) = -x + \mathcal{O}(x^2)$ blèpoume ìti o pr¸toc ìroc eÐnai thc t^xhc e^n ìpwc kai oi upìloipoi.

Epomènwc gr^fontac ton pÐnaka puknìthtac sthn morf 6.2.1 exasfalÐzoume ìti h entropÐa sÔmplexh eÐnai thc t^xhc e^n [2] kai ètsi xèroume mèqri poiac t^xhc ìrouc prèpei na krat soume.

Gia na efarmìsoume autì sthn perÐptwsh pou melet^me arqik^ parathroÔme ìti oi 2 pr¸tec grammèc thc exÐswshc 6.1.2 gr^fontai sthn morf $|\Psi\rangle\langle\Psi|$, ìpou

$$|\Psi\rangle = |\alpha\rangle_H + C_\beta|\beta\rangle_H + \sum_{\omega_\gamma > E_d} C_{\beta\gamma}|\beta\gamma\rangle_H + \ldots \tag{6.2.4}$$

sun ìrouc pou den perilamb^noun thn kat^stash $|\alpha\rangle_H$. Oi suntelestèc C_β eÐnai thc t^xhc e^N, me thn mikrìterh tim tou N na eÐnai 2. Oi upìloipoi suntelestèc eÐnai an¸terhc t^xhc. Epomènwc ston pÐnaka puknìthtac ìroi pou h mia apì tic 2 katast^seic ket-bra den eÐnai h $|\alpha\rangle_H$ eÐnai t^xhc e^{2N} megalÔterhc.

Me b^sh thn èkfrash 6.2.4 mporoÔme na qwrÐsoume ton pÐnaka puknìthtac se 2 mèrh, ìpou to èna mèroc na eÐnai thc t^xhc e^{2N} megalÔterhc. K^nontac èna sÔntomo upologismì, brÐskoume ìti o pÐnakac puknìthtac paÐrnei th morf

$$\rho_H = |\Psi\rangle\langle\Psi| + |\beta\rangle\langle\beta|G \tag{6.2.5}$$

[2] Υπάρχει το ενδεχόμενο όλοι οι όροι τάξης e^n να αναιρούνται και η εντροπία σύμπλεξης να είναι μεγαλύτερης τάξης. Σε αυτή την περίπτωση πρέπει να κρατήσουμε όρους μεγαλύτερης τάξης.

ìpou

$$
\begin{aligned}
G = \tilde{A}_{\beta\alpha}\tilde{A}^{*}_{\beta'\alpha}\langle f_{\beta'}|f_{\beta}\rangle - \tilde{A}_{\beta\alpha}\tilde{A}^{*}_{\beta'\alpha}\langle f_{\beta'}|f_{\alpha}\rangle\langle f_{\alpha}|f_{\beta}\rangle \\
\sum_{\omega_\gamma<E_d}\frac{1}{(2V\omega_\gamma)^{\frac{1}{2}}}\tilde{B}_{\beta\gamma,\alpha}\tilde{A}^{*}_{\beta'\alpha}\left(f^{*}_{\beta'}(\vec{q}_\gamma)-f^{*}_{\beta}(\vec{q}_\gamma)\right)\cdot\epsilon(\gamma)\langle f_{\beta'}|f_{\beta}\rangle \\
+\sum_{\omega'_\gamma<E_d}\frac{1}{(2V\omega_\gamma)^{\frac{1}{2}}}\tilde{B}^{*}_{\beta'\gamma',\alpha}\tilde{A}_{\beta\alpha}\left(f_{\beta}(\vec{q}_{\gamma'})-f_{\beta'}(\vec{q}_{\gamma'})\right)\cdot\epsilon^{*}(\gamma')\langle f_{\beta'}|f_{\beta}\rangle \\
-\sum_{\omega_\gamma<E_d}\frac{1}{(2V\omega_\gamma)^{\frac{1}{2}}}\tilde{B}_{\beta\gamma,\alpha}\tilde{A}^{*}_{\beta'\alpha}\left(f^{*}_{\alpha}(\vec{q}_\gamma)-f^{*}_{\beta}(\vec{q}_\gamma)\right)\cdot\epsilon(\gamma)\langle f_{\alpha}|f_{\beta}\rangle \\
-\sum_{\omega'_\gamma<E_d}\frac{1}{(2V\omega_\gamma)^{\frac{1}{2}}}\tilde{B}^{*}_{\beta'\gamma',\alpha}\tilde{A}_{\beta\alpha}\left(f_{\alpha}(\vec{q}_{\gamma'})-f_{\beta'}(\vec{q}_{\gamma'})\right)\cdot\epsilon^{*}(\gamma')\langle f_{\alpha}|f_{\beta}\rangle \\
+\sum_{\omega_\gamma<E}\tilde{B}_{\beta\gamma,\alpha}\tilde{B}^{*}_{\beta'\gamma,\alpha}\langle f_{\beta'}|f_{\beta}\rangle+\ldots
\end{aligned}
\tag{6.2.6}
$$

Oi pr¸toi dÔo ìroi eÐnai t^xhc e^{2N} en¸ oi upìloipoi ìroi eÐnai thc t^xhc e^{2N+2}. Qrhsimopoi¸ntac thn sqèsh

$$
\langle f_\beta|f_\alpha\rangle = \left(\frac{\lambda}{E_d}\right)^{\mathcal{B}_{\alpha\beta}} = 1+\mathcal{B}_{\alpha\beta}\log\frac{\lambda}{E_d}+\ldots \tag{6.2.7}
$$

parathroÔme ìti ta mèrh twn dÔo pr¸twn ìrwn pou eÐnai thc t^xhc e^{2N} anairoÔntai, kai ètsi to ˆjroisma eÐnai thc t^xhc e^{2N+2}, ìpwc kai oi upìloipoi ìroi. Katafèrame na gr^youme ton pÐnaka puknìthtac sthn morf 6.2.1 [3] me $n = 2N+2$. H entropÐa sÔmplexhc eÐnai thc t^xhc e^{2N+2}, me $N_{min} = 2$. Gia ton upologismì thc kÔriac entropÐac sÔmplexhc ameloÔme ìrouc megalÔterhc t^xhc. EpÐshc axÐzei na shmeiwjeÐ ìti h èkfrash 6.2.5 deÐqnei ìti o pÐnakac puknìthtac eÐnai diag¸nioc kat^ mèrh kai ètsi h entropÐa sÔmplexhc den eÐnai mhdèn se t^xh e^{2N+2}.

Apì ton pio p^nw upologismì agno same telikèc katast^seic me dÔo perissìtera ekpempìmena fwtìnia. Gia na doÔme giatÐ ac sumperil^boume katast^seic me dÔo ekpempìmena fwtìnia wc akoloÔjwc

$$
|\beta\gamma_1\gamma_2\rangle \tag{6.2.8}
$$

H telik kat^stash èqei th morf

$$
|\Psi\rangle_{out} = |\alpha\rangle_d + \tilde{A}_{\beta\alpha}|\beta\rangle_d + \tilde{B}_{\beta\gamma,\alpha}|\beta\gamma\rangle_d + \tilde{\Gamma}_{\beta\gamma_1\gamma_2,\alpha}|\beta\gamma_1\gamma_2\rangle + \ldots \tag{6.2.9}
$$

ìpou $\tilde{\Gamma}_{\beta\gamma_1\gamma_2}$ eÐnai thc t^xhc e^{N+2}. Qrhsimopoi¸ntac an^logec sqèseic me tic 5.7.2–5.7.8, brÐskoume ìti oi ìroi sthn exÐswsh 6.1.2 pou perilamb^noun katast^seic $|\beta\gamma_1\gamma_2\rangle$ kai $|\beta\gamma\rangle$ $|\beta\rangle$ pollaplasi^zontai me èna suntelest pou eÐnai ginìmeno twn plat¸n $\tilde{\Gamma}_{\beta\gamma_1\gamma_2,\alpha}$ kai

[3] Για να είναι αυτής της μορφής πρέπει να κανονικοποιήσουμε την $|\Psi\rangle$, αλλά αυτό δεν επηρεάζει τα συμπεράσματα.

$\tilde{B}_{\beta\gamma,\alpha}$ $\tilde{A}_{\beta\alpha}$ kai me mia sun^rthsh an^logh tou $f(\vec{q}_\gamma)$. AfoÔ h sun^rthsh $f(\vec{q}_\gamma)$ eÐnai an^logh tou fortÐou tou hlektronÐou e, autoÐ oi ìroi eÐnai thc t^xhc e^{2N+3} toul^qisto. Oi ìroi pou perilamb^noun tic katast^seic $|\beta\gamma_1\gamma_2\rangle$ kai $|\alpha\rangle$ mporoÔn na aporrofhjoÔn sthn kat^stash $|\Psi\rangle$ (ìpwc orÐsthke sthn sqèsh 6.2.5). O lìgoc eÐnai ìti to mètro thc kat^stashc $|\Psi\rangle$ perilamb^nei katast^seic me dÔo (perissìtera) fwtìnia se ìrouc pou perièqoun èna par^gonta $\tilde{\Gamma}_{\beta\gamma_1\gamma_2}$, èna par^gonta $\tilde{B}_{\beta\gamma,\alpha}$ $\tilde{A}_{\beta\alpha}$ kai mia sun^rthsh an^logh tou $f(\vec{q}_\gamma)$. Epomènwc eÐnai thc t^xhc e^{2N+3} kai den suneisfèroun sthn entropÐa sÔmplexhc se t^xh e^{2N+2}. Oi upìloipoi ìroi pou perilamb^noun th kat^stash $|\beta\gamma_1\gamma_2\rangle$ eÐnai an¸terhc t^xhc. 'Ara loipìn katast^seic me dÔo perissìtera ekpempìmena fwtìnia den suneisfèroun sthn kÔria t^xh thc entropÐac sÔmplexhc.

6.3 Diataraktik an^lush

Stouc akìloujouc upologismoÔc ja krat soume ìrouc t^xhc e^{2N+2}. H entropÐa sÔmplexhc dÐnetai apì thn èkfrash

$$S_{ent} = -\,\mathrm{Tr}\,\rho_H \log \rho_H \tag{6.3.1}$$

Gia na qeiristoÔme ton log^rijmo tou pÐnaka puknìthtac, gr^foume thn entropÐa sÔmplexhc sthn morf

$$S_{ent} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{(n-1)!}{(n-m)!(m)!}(-1)^m e^{-mS_{m+1}} \tag{6.3.2}$$

ìpou

$$S_m = \frac{1}{1-m}\log \mathrm{Tr}(\rho_H)^m \tag{6.3.3}$$

oi entropÐec Renyi. Ja upologÐsoume tic entropÐec Renyi gia akèraia $m \geq 2$ mèqri thn t^xh e^{2N+2} [4].

Jètontac

$$\rho_H = \rho_0 + \epsilon, \quad \rho_0 = |\alpha\rangle_H \langle\alpha|_H \tag{6.3.4}$$

[4] Το άθροισμα στην σχέση 6.3.2 μηδενίζεται εάν αντικαταστήσουμε το εκθετικό με την μονάδα. Επομένως στην κύρια τάξη της εντροπίας σύμπλεξης συνεισφέρει ο πρώτος όρος στο ανάπτυγμα του εκθετικού που είναι της τάξης των εντροπιών Renyi.

kai qrhsimopoi¸ntac tic sqèseic $\mathrm{Tr}\,\rho_H = 1 = \mathrm{Tr}\,\rho_0$, paÐrnoume $\mathrm{Tr}\,\epsilon = 0$. Pr^gmati upologÐzontac to Ðqnoc thc diataraq c ϵ brÐskoume

$$\mathrm{Tr}\,\epsilon = C_\alpha + C^*_\alpha + \sum_\beta \left(D_{\beta,\beta} + \sum_{\omega_\gamma > E} D_{\beta\gamma,\beta\gamma} \right) + \dots$$
$$= \tilde{A}_{\alpha\alpha} + \tilde{A}^*_{\alpha\alpha} + \sum_\beta \tilde{A}_{\beta\alpha}\tilde{A}^*_{\beta\alpha} + \sum_{\beta\gamma} \tilde{B}_{\beta\gamma,\alpha}\tilde{B}^*_{\beta\gamma,\alpha} + \dots =_d \langle \alpha | i(T - T^\dagger) + T^\dagger T | \alpha \rangle_d = 0 \tag{6.3.5}$$

me b^sh tic sqèseic 6.3 kai 6.4. EpÐshc qrhsimopoi same tic sqèseic orjogwniìthtac [5], $\langle \beta_1 | \beta_2 \rangle = \delta_{\beta_1\beta_2}$, ìpou oi deÐktec β_1, β_2 paÐrnoun timèc $\alpha, \beta, \beta', \beta\gamma \dots$. SumperaÐnoume loipìn ìti to ^jroisma twn dÔo pr¸twn ìrwn thc pio p^nw èkfrashc eÐnai thc t^xhc e^{2N}.

H diataraq ε eÐnai thc t^xhc e^N kai jèloume na krat soume ìrouc t^xhc e^{2N+2} sta Ðqnh $\mathrm{Tr}(\rho_H)^m$. Gia $N \geq 2$ ja prèpei na anaptÔxoume ton pÐnaka ρ_H^m mèqri kubikoÔc ìrouc wc proc ε. Epeid $\rho_0^2 = \rho_0$ kai lìgw thc kuklik idiìthta tou Ðqnouc, arkeÐ na upologÐsoume èna sqetik^ mikrì arijmì anex^rthtwn ìrwn, wc akolouÔjwc:

Stouc grammikoÔc ìrouc wc proc ϵ arkeÐ na upologÐsoume ton ìro $\epsilon\rho_0$, tou opoÐou to Ðqnoc isoÔtai me

$$\mathrm{Tr}\,\epsilon\rho_0 = C_\alpha + C^*_\alpha + D_{\alpha,\alpha} = \tilde{A}_{\alpha\alpha} + \tilde{A}^*_{\alpha\alpha} + \tilde{A}_{\alpha\alpha}\tilde{A}^*_{\alpha\alpha} + \sum_{\omega_\gamma > E} \tilde{B}_{\beta\gamma,\alpha}\tilde{B}^*_{\beta\gamma,\alpha} + \dots \tag{6.3.6}$$

Oi teleÐec perilamb^noun ìrouc t^xhc megalÔterhc apì e^{2N+2} kai den suneisfèroun sthn kÔria t^xh thc entropÐac sÔmplexhc.

Stouc tetragwnikoÔc ìrouc perilamb^nontai oi ìroi $\epsilon^2, \epsilon^2\rho_0, \epsilon\rho_0\epsilon\rho_0$. To Ðqnoc tou pr¸tou ìrou eÐnai

$$\mathrm{Tr}\,\epsilon^2 = C^2_\alpha + C^{*2}_\alpha + \sum_\beta \left(2D_{\alpha,\beta}C_\beta + 2D_{\beta,\alpha}C^*_\beta + 2C_\beta C^*_\beta + 2\sum_{\omega_\gamma > E} C_{\beta\gamma,\alpha}C^*_{\beta\gamma,\alpha} \right) + \dots$$
$$= \tilde{A}^2_{\alpha\alpha} + \tilde{A}^{*2}_{\alpha\alpha} + 2\sum_\beta |\langle f_\beta | f_\alpha \rangle|^2 \left(1 + \tilde{A}_{\alpha\alpha} + \tilde{A}^*_{\alpha\alpha}\right) \tilde{A}_{\beta\alpha}\tilde{A}^*_{\beta\alpha} + 2\sum_\beta \sum_{\omega_\gamma < E_d} \frac{|\langle f_\beta | f_\alpha \rangle|^2}{(2V\omega_\gamma)^2}$$
$$\times \left(\tilde{A}_{\beta\alpha}\tilde{B}^*_{\beta\gamma,\alpha} \left(f_\alpha(\vec{q}_\gamma) - f_\beta(\vec{q}_\gamma) \right) \cdot \epsilon^*(\gamma) + \tilde{A}^*_{\beta\alpha}\tilde{B}_{\beta\gamma,\alpha} \left(f^*_\alpha(\vec{q}_\gamma) - f^*_\beta(\vec{q}_\gamma) \right) \cdot \epsilon(\gamma) \right)$$
$$+ 2\sum_\beta \sum_{\omega_\gamma > E} |\langle f_\beta | f_\alpha \rangle|^2 \tilde{B}_{\beta\gamma,\alpha}\tilde{B}^*_{\beta\gamma,\alpha} + \dots \tag{6.3.7}$$

H trÐth gramm mhdenÐzetai gia sumbatikèc katast^seic sth b^sh Fock.

[5] Επίσης παρατηρούμε αυτήν την σχέση από την σχέση κανονικοποίησης της τελικής κατάστασης.

Gia ton deÔtero ìro br’skoume

$$\begin{aligned}
\operatorname{Tr}\epsilon^2\rho_0 &= C_\alpha^2 + C_\alpha^{*2} + C_\alpha C_\alpha^* + D_{\alpha,\alpha}(C_\alpha + C_\alpha^*) \\
&+ \sum_\beta \left(D_{\alpha,\beta}C_\beta + D_{\beta,\alpha}C_\beta^* + C_\beta C_\beta^* + \sum_{\omega_\gamma > E} C_{\beta\gamma,\alpha}C_{\beta\gamma,\alpha}^* \right) + \dots \\
&= \tilde{A}_{\alpha\alpha}^2 + \tilde{A}_{\alpha\alpha}^{*2} + \left(1 + \tilde{A}_{\alpha\alpha} + \tilde{A}_{\alpha\alpha}^*\right) \tilde{A}_{\alpha\alpha}\tilde{A}_{\alpha\alpha}^* + \sum_\beta |\langle f_\beta | f_\alpha \rangle|^2 \left(1 + \tilde{A}_{\alpha\alpha} + \tilde{A}_{\alpha\alpha}^*\right) \tilde{A}_{\beta\alpha}\tilde{A}_{\beta\alpha}^* \\
&+ \sum_\beta \sum_{\omega_\gamma < E_d} \frac{|\langle f_\beta | f_\alpha \rangle|^2}{(2V\omega_\gamma)^2} \Big(\tilde{A}_{\beta\alpha}\tilde{B}_{\beta\gamma,\alpha}^* \left(f_\alpha(\vec{q}_\gamma) - f_\beta(\vec{q}_\gamma) \right) \cdot \epsilon^*(\gamma) \\
&+ \tilde{A}_{\beta\alpha}^*\tilde{B}_{\beta\gamma,\alpha} \left(f_\alpha^*(\vec{q}_\gamma) - f_\beta^*(\vec{q}_\gamma) \cdot \epsilon(\gamma) \right) \Big) + \sum_\beta \sum_{\omega_\gamma > E} |\langle f_\beta | f_\alpha \rangle|^2 \tilde{B}_{\beta\gamma,\alpha}\tilde{B}_{\beta\gamma,\alpha}^* + \dots \qquad (6.3.8)
\end{aligned}$$

Tèloc pa’rnoume

$$\begin{aligned}
\operatorname{Tr}\epsilon\rho_0\epsilon\rho_0 &= (C_\alpha + C_\alpha^*)(C_\alpha + C_\alpha^* + 2D_{\alpha,\alpha}) + \dots \\
&= (\tilde{A}_{\alpha\alpha} + \tilde{A}_{\alpha\alpha}^*)(\tilde{A}_{\alpha\alpha} + \tilde{A}_{\alpha\alpha}^* + \tilde{A}_{\alpha\alpha}\tilde{A}_{\alpha\alpha}^*) + \dots \qquad (6.3.9)
\end{aligned}$$

To ’qnoc autì den suneisfèrei se t^xh e^{2N+2} lìgw monadiakìthtac.

Oi anex^rthtoi kubiko’ ìroi wc proc ε e’nai oi ϵ^3, $\epsilon^3\rho_0$, $\epsilon^2\rho_0\epsilon\rho_0$, $\epsilon\rho_0\epsilon\rho_0\epsilon\rho_0$. Pr¸ta br’skoume

$$\begin{aligned}
\operatorname{Tr}\epsilon^3 &= C_\alpha^3 + C_\alpha^{*3} + 3(C_\alpha + C_\alpha^*) \sum_\beta C_\beta C_\beta^* + \dots \\
&= \tilde{A}_{\alpha\alpha}^3 + \tilde{A}_{\alpha\alpha}^{*3} + 3(\tilde{A}_{\alpha\alpha} + \tilde{A}_{\alpha\alpha}^*) \sum_\beta C_\beta C_\beta^* + \dots \qquad (6.3.10)
\end{aligned}$$

to opo’o den suneisfèrei se t^xh e^{2N+2} lìgw monadiakìthtac. Ta upìloipa ’qnh d’dontai apì tic sqèseic

$$\operatorname{Tr}\bigl(\epsilon^3\rho_0\bigr) = C_\alpha^3 + C_\alpha^{*3} + (C_\alpha + C_\alpha^*) \left(C_\alpha C_\alpha^* + \sum_\beta C_\beta C_\beta^* \right) + \dots \qquad (6.3.11)$$

$$\operatorname{Tr}\bigl(\epsilon^2\rho_0\epsilon\rho_0\bigr) = (C_\alpha + C_\alpha^*) \left(C_\alpha^2 + C_\alpha^{*2} + C_\alpha C_\alpha^* + \sum_\beta C_\beta C_\beta^* \right) + \dots \qquad (6.3.12)$$

$$\operatorname{Tr}(\epsilon\rho_0\epsilon\rho_0\epsilon\rho_0) = (C_\alpha + C_\alpha^*)^3 + \dots \qquad (6.3.13)$$

kai den suneisfèroun sthn kÔria t^xh thc entrop’ac sÔmplexhc.

'Ara loipìn, mìno ta Ðqnh $\mathrm{Tr}(\epsilon\rho_0)$, $\mathrm{Tr}(\epsilon^2)$ kai $\mathrm{Tr}(\epsilon^2\rho_0)$ suneisfèroun sthn kÔria t^xh thc entropÐac sÔmplexhc.

Gia touc upologismoÔc pou akoloujoÔn, ja qreiastoÔme thn sqèsh

$$\langle f_\beta | f_\alpha \rangle = \left(\frac{\lambda}{E_d}\right)^{\mathcal{B}_{\alpha\beta}} = 1 + \mathcal{B}_{\alpha\beta} \log \frac{\lambda}{E_d} + \dots \tag{6.3.14}$$

ìpou $\mathcal{B}_{\alpha\beta}$ eÐnai thc t^xhc e^2. UpenjumÐzoume ìti gia touc diataraktikoÔc upologismoÔc krat^me ton ìgko tou koutioÔ, kai epomènwc to ìrio apokop c, peperasmèno, kai exet^-zoume to ìrio $\lambda \to 0$ sto tèloc. EpÐshc ja qreiastoÔme tic sqèseic

$$\begin{aligned}
\tilde{S}_{\beta\alpha} &= \left(\frac{E_d}{\lambda}\right)^{\mathcal{B}_{\alpha\beta}} S_{\beta\alpha} \\
\tilde{S}_{\beta\gamma,\alpha} &= \left(\frac{E_d}{\lambda}\right)^{\mathcal{B}_{\alpha\beta}} S_{\beta\gamma,\alpha}, \quad \omega_\gamma > E_d \\
\tilde{B}_{\beta\gamma,\alpha} &= \frac{1}{(2V^5\omega_\gamma)^{\frac{1}{2}}} F_{\alpha\beta}(\vec{q}_\gamma, \epsilon_r(\vec{q}_\gamma)), \quad \omega_\gamma < E_d
\end{aligned} \tag{6.3.15}$$

Sthn kÔria t^xh e^{N+1} h sun^rthsh $F_{\alpha\beta}(\vec{q}_\gamma, \epsilon_r(\vec{q}_\gamma))$ den apoklÐnei sto ìrio $\lambda, |\vec{q}_\gamma| \to 0$. H ex^rthsh apì ton ìgko ofeÐletai sthn sqetik kanonikopoÐhsh metaxÔ twn katast^sewn tou koutioÔ kai twn katast^sewn tou ^peirou q¸rou.

Me b^sh tic pio p^nw sqèseic susqetÐzoume ta pl^th twn ntumènwn katast^sewn me aut^ twn gumn¸n katast^sewn

$$\tilde{A}_{\beta\alpha} = \frac{A_{\beta\alpha}}{\langle f_\beta | f_\alpha \rangle} \tag{6.3.16}$$

kai

$$\tilde{B}_{\beta\gamma,\alpha} = \frac{B_{\beta\gamma,\alpha}}{\langle f_\beta | f_\alpha \rangle} \tag{6.3.17}$$

Ja ekfr^soume ta Ðqnh sunart sei twn sumbatik¸n plat¸n skèdashc sth b^sh Fock, ta opoÐa eÔkola mporoÔme na upologÐsoume sunart sei diagramm^twn Feynman.

Gia to grammikì Ðqnoc paÐrnoume

$$\mathrm{Tr}\,\epsilon\rho_0 = A_{\alpha\alpha} + A^*_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha} + \left(\sum_{\omega_\gamma < E_d} \frac{1}{2V^5\omega_\gamma} F_{\alpha\alpha}(\gamma) F^*_{\alpha\alpha} + \sum_{E_d < \omega_\gamma < E} B_{\alpha\gamma,\alpha} B^*_{\alpha\gamma,\alpha} \right) \tag{6.3.18}$$

O pr¸toc ìroc sthn parènjesh dÐdetai sto suneqèc ìrio apì thn èkfrash

$$\int_\lambda^{E_d} \frac{d^3q}{(2\pi)^3 2\omega_{\vec{q}}} \sum_r |F_{\alpha\alpha}(\vec{q}_\gamma, \epsilon_r(\vec{q}_\gamma))|^2 \tag{6.3.19}$$

Epeid h sun^rthsh $F_{\alpha\alpha}(\vec{q}_\gamma, \epsilon_r(\vec{q}_\gamma))$ eÐnai omal sto ìrio $|\vec{q}|, \gamma \to 0$, h suneisfor^ tou ìrou autoÔ eÐnai thc t^xhc E_d^2 kai mporeÐ na amelhjeÐ [1].

Gia ta tetragwnik^ ìqnh paÐrnoume

$$
\begin{aligned}
\operatorname{Tr}\epsilon^2 = A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + 2\sum_\beta \left(A_{\beta\alpha}A_{\beta\alpha}^* + \sum_{\omega_\gamma > E} B_{\beta\alpha,\alpha}B_{\beta\gamma,\alpha}^* \right) \\
+2\sum_\beta \sum_{\omega_\gamma < E_d} \frac{1}{2V^3\omega_\gamma} \\
\times\Big(A_{\beta\gamma}F_{\beta\alpha}^*(\gamma)\left(f_\alpha(\vec{q}_\gamma) - f_\beta(\vec{q}_\gamma)\right)\cdot\epsilon^*(\gamma) + A_{\beta\alpha}^*F_{\beta\alpha}\left(f_\alpha^*(\vec{q}_\gamma) - f_\beta^*(\vec{q}_\gamma)\cdot\epsilon(\gamma)\right)\Big)
\end{aligned}
\tag{6.3.20}
$$

$$
\begin{aligned}
\operatorname{Tr}\big(\epsilon^2\rho_0\big) = A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + A_{\alpha\alpha}A_{\alpha\alpha}^* + 2\sum_\beta \left(A_{\beta\alpha}A_{\beta\alpha}^* + \sum_{\omega_\gamma > E} B_{\beta\gamma,\alpha}B_{\beta\gamma,\alpha}^* \right) \\
+2\sum_\beta \sum_{\omega_\gamma < E_d} \frac{1}{2V^3\omega_\gamma} \\
\times\Big(A_{\beta\alpha}F_{\beta\alpha}^*(\gamma)\left(f_\alpha(\vec{q}_\gamma) - f_\beta(\vec{q}_\gamma)\right)\cdot\epsilon^*(\gamma) + A_{\beta\alpha}^*F_{\beta\alpha}\left(f_\alpha^*(\vec{q}_\gamma) - f_\beta^*(\vec{q}_\gamma)\cdot\epsilon(\gamma)\right)\Big)
\end{aligned}
\tag{6.3.21}
$$

Epeid h sun^rthsh $F_{\beta\alpha}$ eÐnai thc t^xhc e^{N+1} kai h f_α eÐnai an^logh tou fortÐou e, arkeÐ to pl^toc $A_{\beta\alpha}$ na upologisteÐ sthn prosèggish diagr^mmatoc dèntrou dhlad se t^xh e^N. Oi ekfr^seic stic parenjèseic eÐnai thc t^xhc e^{2N+2}. Epeid se aut thn t^xh, sto pl^toc $A_{\beta\alpha}$ den suneisfèroun diagr^mmata brìqou, den prokÔptoun apì autì logarijmikoÐ apeirismoÐ sto ìrio $\lambda \to 0$.

Sto suneqèc ìrio, oi teleutaÐec grammèc stic ekfr^seic 6.3.20 kai 6.3.21 dÐnontai apì to pio k^tw olokl rwma

$$
\int_\lambda^{E_d} \frac{d^3q}{(2\pi)^3 2\omega_{\vec{q}}} \sum_r F_{\beta\alpha}^*(\vec{q}_\gamma, \epsilon_r(\vec{q}_\gamma)) \sum_{s\in\{\alpha,\alpha\}} \frac{e_s\eta_s p_s\cdot\epsilon_r^*(\vec{q})}{p_s\cdot q} + h.c. \tag{6.3.22}
$$

To olokl rwma den parousi^zei apeirismì sto ìrio $|\vec{q}| \to 0$, kai eÐnai thc t^xhc E_d me amelhtèa suneisfor^ [1].

T¸ra mporoÔme na upologÐsoume to ìqnoc $\operatorname{Tr}(\rho_H)^2$ se t^xh e^{2N+2}.

$$
\begin{aligned}
\operatorname{Tr}(\rho_H)^2 = \operatorname{Tr}(\rho_0)^2 + 2\operatorname{Tr}\epsilon\rho_0 + \operatorname{Tr}\epsilon^2 \\
= 1 + 2(A_{\alpha\alpha} + A_{\alpha\alpha}^*) + (A_{\alpha\alpha} + A_{\alpha\alpha}^*)^2 + 2\sum_\beta \left(A_{\beta\alpha}A_{\beta\alpha}^* + \sum_{\omega_\gamma > E} B_{\beta\gamma,\alpha}B_{\beta\gamma,\alpha}^* \right)
\end{aligned}
\tag{6.3.23}
$$

Lìgw monadiakìthtac, o trÐtoc ìroc eÐnai thc t^xhc e^{2N+4} kai ton ameloÔme.

6.4 KÔria t^xh entropÐac sÔmplexhc

Qrhsimopoi¸ntac thn sqèsh plhrìthtac 6.3.5, h èkfrash 6.3.23 aplopoieÐtai sthn

$$\mathrm{Tr}(\rho_H)^2 = 1 - 2\Delta \tag{6.4.1}$$

ìpou

$$\Delta = \sum_{\beta} \sum_{\omega_\gamma < E} B_{\beta\gamma,\alpha} B^*_{\beta\gamma,\alpha} \tag{6.4.2}$$

mia posìthta thc t^xhc e^{2N+2}, h opoÐa exart^tai mìno apì ta sumbatik^ pl^th ekpomp c enìc fwtonÐou me enèrgeia $\lambda < \omega_\gamma < E$.

AkoloÔjwc ja upologÐsoume ta Ðqnh $\mathrm{Tr}(\rho_H)^m$ me $m \geq 3$ se t^xh e^{2N+2}. Se aut thn t^xh mporoÔme na qrhsimopoi soume thn èkfrash

$$\mathrm{Tr}(\rho_H)^m = 1 + m\,\mathrm{Tr}\,\epsilon\rho_0 + m\,\mathrm{Tr}\,\epsilon^2\rho_0 \tag{6.4.3}$$

Me b^sh thn pio p^nw èkfrash katal goume sth sqèsh

$$\mathrm{Tr}(\rho_H)^m = 1 - m\Delta \tag{6.4.4}$$

Oi entropÐec Renyi paÐrnoun thn morf

$$S_{m+1} = -\frac{1}{m}\log[1-(m+1)\Delta] = \frac{m+1}{m}\Delta \tag{6.4.5}$$

ìpou amel same ìrouc thc t^xhc Δ^2. Qrhsimopoi¸ntac tic sqèseic 6.3.2, 6.4.5 kai thn tautìthta

$$\sum_{m=0}^{n} = \binom{n}{m}(-1)^m = 0 \tag{6.4.6}$$

katal goume sthn telik sqèsh [6]

$$S_{ent} = \Delta \tag{6.4.7}$$

H posìthta Δ apoklÐnei sto ìrio $\lambda \to 0$. Exet^zoume to apoklÐnon mèroc, to opoÐo kuriarqeÐ sthn entropÐa sÔmplexhc

$$S_{ent} = \Delta = \sum_{\beta} \sum_{\omega_\gamma < E_d} B_{\beta\gamma,\alpha} B^*_{\beta\gamma,\alpha} \tag{6.4.8}$$

[6] Το διπλό άθροισμα ως προς n και m μηδενίζεται για όλα τα $n \neq 1$.

ìpou amel same th suneisfor^ fwtonÐwn me enèrgeia $E_d < \omega_\gamma < E$ sto ^jroisma. Qrhsimopoi¸ntac ta upèrujra jewr mata gia fwtìnia, paÐrnoume

$$S_{ent} = \sum_\beta (A_{\beta\alpha} A^*_{\beta\alpha}) \left(\sum_{\omega_\gamma < E_d} \frac{1}{2V\omega_\gamma} \sum_{s,s' \in \{\alpha,\beta\}} e_s e_{s'} \eta_s \eta_{s'} \frac{p_s p_{s'}}{(p_s q_\gamma)(p_{s'} q_\gamma)} \right) \tag{6.4.9}$$

ìpou ta pl^th $A_{\beta\alpha}$ upologÐzontai se diagr^mmata dèntrou. Ta diagr^mmata pou suneisfèroun eÐnai asÔndeta kai èqoun toul^qiston dÔo korufèc. Epomènwc h kÔria t^xh thc entropÐac sÔmplexhc eÐnai e^6 [7].

6.5 Suneqèc ìrio

Sto suneqèc ìrio prèpei na l^boume upìyin th sqetik kanonikopoÐhsh metaxÔ twn katast^sewn sto koutÐ kai twn katast^sewn ston ^peiro q¸ro

$$|\vec{p}\rangle_{Box} = \frac{1}{(2E_{\vec{p}}V)^{1/2}} |\vec{p}\rangle \tag{6.5.1}$$

Me b^sh tic ekfr^seic gia ta pl^th skèdashc $A_{\beta\alpha}$ (dec par^rthma C) paÐrnoume

$$\begin{aligned} S_{ent,sing} = {} & \frac{1}{V^I} \sum_\vartheta \frac{1}{n_\vartheta! m_\vartheta! l_\vartheta!} \\ & \times \Bigg\{ \left(\prod_f \int \frac{d^3 \vec{p}_f}{(2\pi)^3 2E_f} \right) \left(\prod_{i=1}^{I} \frac{1}{2E_i} \right) \Big| i\mathcal{M}(p_i \to p_f) \Big|^2 \left[(2\pi)^4 \delta^4 \Big(\sum_f p_f - \sum_i p_i \Big) \right]^2 \\ & \times \int_\lambda^{E_d} \frac{d^3 \vec{q}}{(2\pi)^3 2\omega_{\vec{q}}} \sum_{s,r \in \{\alpha,\vartheta\}} \eta_s \eta_r e_s e_r \frac{p_s p_r}{(p_s q)(p_r q)} \Bigg\} \end{aligned} \tag{6.5.2}$$

ìpou n_ϑ, m_ϑ kai l_ϑ o arijmìc twn hlektronÐwn, pozitronÐwn kai fwtonÐwn antÐstoiqa sthn telik kat^stash ϑ oloklhr¸mata [8]. To pl^toc iM upologÐzetai se diagr^mmata dèntrou. EpÐshc I eÐnai o sunolikìc arijmìc twn arqik¸n swmatidÐwn kai k^je arqikì swmatÐdio suneisfèrei me èna par^gonta $1/V$ exaitÐac thc sqetik c kanonikopoÐhshc.

Qrhsimopoi¸ntac thn sqèsh

$$\int_\lambda^{E_d} \frac{d^3 \vec{q}}{(2\pi)^3 2\omega_{\vec{q}}} \sum_{s,r \in \{\alpha,\vartheta\}} \eta_s \eta_r e_s e_r \frac{p_s p_r}{(p_s q)(p_r q)} = \log\left(\frac{E_d}{\lambda}\right) (2\mathcal{B}_{\vartheta\alpha}(p_f)) \tag{6.5.3}$$

[7] Για να φτάσουμε σε αυτό το αποτέλεσμα υποθέσαμε ότι το εσωτερικό γινόμενο μεταξύ των καταστάσεων $|\beta\gamma\rangle$ και $|\alpha\rangle$ είναι μηδέν ($\langle\beta\gamma|\alpha\rangle = 0$). Στην περίπτωση που έχουμε φωτόνια στην αρχική κατάσταση η σχέση $\langle\beta\gamma|\alpha\rangle = 0$ ισχύει για περιπτώσεις που ο συντελεστής $A_{\beta\alpha}$ είναι τάξης e^2. Ο λόγος είναι ότι δεν υπάρχουν διαγράμματα μέχρι αυτή την τάξη που να επιτρέπουν απορρόφηση ενός μόνο φωτονίου. Επομένως στην περίπτωση της κύριας τάξης της εντροπίας σύμπλεξης το αποτέλεσμα δεν θα άλλαζε αν επιτρέπαμε στην αρχική κατάσταση να έχει φωτόνια.

[8] Το σύμβολο $\sum_\vartheta$ συμβολίζει άθροισμα ως προς τα διαφορετικά σωματίδια που μπορεί να έχει η τελική κατάσταση.

ìpou $\mathcal{B}_{\vartheta\alpha}(\vec{p}_f) = \mathcal{B}_{\beta\alpha}$ kai $\vec{p}_f$ sumbolÐzei tic ormèc twn swmatidÐwn thc telik c kat^stashc, gr^foume thn èkfrash 6.5.2 sthn morf

$$S_{ent,sing} = \frac{2}{V^I}\log\left(\frac{E_d}{\lambda}\right)\sum_{\vartheta}\frac{1}{n_\vartheta! m_\vartheta! l_\vartheta!}$$
$$\times\left\{\prod_f\left(\int\frac{d^3\vec{p}_f}{(2\pi)^3 2E_f}\right)\left(\prod_{i=1}^{I}\frac{1}{2E_i}\right)\Big|i\mathcal{M}(p_i\to p_f)\Big|^2\mathcal{B}_{\vartheta\alpha}\Big[(2\pi)^4\delta^4\Big(\sum_f p_f-\sum_i p_i\Big)\Big]^2\right\} \tag{6.5.4}$$

AkoloÔjwc ja gr^foume $\mathcal{B}_{\vartheta\alpha}$ antÐ gia $\mathcal{B}_{\vartheta\alpha}(p_f)$.

Sthn perÐptwsh pou sthn arqik kat^stash up^rqoun 2 swmatÐdia (ta i kai j) (kai epomènwc 2 swmatÐdia stic telikèc katast^seic ϑ), h èkfrash paÐrnei thn morf

$$S_{ent,sing} = \frac{2}{V^2}\log\left(\frac{E_d}{\lambda}\right)\sum_{\vartheta}\frac{1}{n_\vartheta! m_\vartheta! l_\vartheta!}$$
$$\times\left\{\prod_f\left(\int\frac{d^3\vec{p}_f}{(2\pi)^3 2E_f}\right)\frac{1}{2E_i 2E_j}\Big|i\mathcal{M}(p_i+p_j\to\sum_f p_f)\Big|^2\mathcal{B}_{\vartheta\alpha}\Big[(2\pi)^4\delta^4\Big(\sum_f p_f-p_i-p_j\Big)\Big]^2 \tag{6.5.5}$$

Ektel¸ntac to olokl rwma wc proc thn telik orm $\vec{p_1}$, epib^lletai diat rhsh thc qwrik c orm c apì thn mia δ^3-sun^rthsh ($\sum_f \vec{p}_f = \vec{p}_i - \vec{p}_j$), kai h deÔterh δ^3-sun^rthsh dÐnei èna par^gonta ìgkou $(2\pi)^3\delta^3(0) = V$. 'Etsi h èkfrash gia thn entropÐa sÔmplexhc an^getai sthn

$$S_{ent,sing} = \frac{2}{V}\log\left(\frac{E_d}{\lambda}\right)\sum_{\vartheta}\frac{1}{n_\vartheta! m_\vartheta! l_\vartheta!}\times\left\{\prod_{f\neq 1}\left(\int\frac{d^3\vec{p}_f}{(2\pi)^3 2E_f}\right)\right.$$
$$\left.\frac{1}{8E_iE_jE_1}\Big|i\mathcal{M}(p_i+p_j\to\sum_f p_f)\Big|^2\mathcal{B}_{\vartheta\alpha}\Big[(2\pi)\delta\Big(\sum_f E_f-E_i-E_j\Big)\Big]^2\right\} \tag{6.5.6}$$

Sto sÔsthma tou kèntrou m^zac, ìpou ta arqik^ kai telik^ swmatÐdia èqoun antÐjetec ormèc, h sqetik taqÔthta twn arqik¸n swmatidÐwn dÐnetai apì thn èkfrash $v_{rel} = 4p_0/E_{cm}$. Oloklhr¸nontac wc proc to mètro thc orm c tou telikoÔ swmatidÐou 2, $|\vec{p}_2|$, epib^lletai diat rhsh thc enèrgeiac kai tautìqrona paÐrnoume èna par^gonta $2\pi\delta(E_i - E_i) = T$, ìpou T o qarakthristikìc qrìnoc tou peir^matoc.

H posìthta v_{rel}/V eÐnai h ro tou arqikoÔ swmatidÐou j wc proc to swmatÐdio i. 'Etsi mporoÔme na orÐsoume thn peperasmènh posìthta $s_{ent,sing}$, pou isoÔtai me thn entropÐa sÔmplexhc an^ mon^da ro c swmatidÐwn an^ mon^da qrìnou.

6.5.1 Plˆtoc iM kÔriac tˆxhc

Sthn perÐptwsh pou h arqik katˆstash perilambˆnei perissìtera apì dÔo swmatÐdia, sthn kÔria tˆxh thc entropÐac sÔmplexhc suneisfèroun asÔndeta diagrˆmmata sta opoÐa mìno dÔo swmatÐdia allhlepidroÔn metaxÔ touc. Kˆje swmatÐdio i me orm $\vec{p}_{i,in}$ pou den allhlepidrˆ suneisfèrei sto diˆgramma Feynman me ènan parˆgonta $(2\pi)^3\, E_{i,in}\, \delta^3(\vec{p}_{i,in} - \vec{p}_{i,out})$. JewroÔme th perÐptwsh ìpou oi ormèc twn arqik¸n swmatidÐwn diafèroun metaxÔ touc.

To plˆtoc metˆbashc iM apì mia arqik katˆstash me n hlektrìnia kai m pozitrìnia se mia telik katˆstash me n hlektrìnia kai m pozitrìnia (qwrÐc fwtìnia) isoÔtai me to ˆjroisma twn akìloujwn tri¸n ìrwn

$$\begin{aligned} I_1 = &\sum_{i=1}^{n}\sum_{j<i}^{n}\sum_{\sigma,\sigma'}\prod_{k\neq i,j}^{n-2}\Big((2\pi)^3 E(\vec{p}_{k,in})\delta^3(\vec{p}_{k,in}-\vec{p}_{\sigma(k),out})\Big)\\ &\times\prod_{k'=1}^{m}\Big((2\pi)^3 E(\vec{k}_{k',in})\delta^3(\vec{k}_{k',in}-\vec{k}_{\sigma'(k'),out})\Big)\\ &\times iM_{1t}\left(\vec{p}_{i,in},\vec{p}_{j,in};\vec{p}_{\sigma(i),out},\vec{p}_{\sigma(j),out}\right)\,\mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma') \end{aligned} \tag{6.5.7}$$

$$\begin{aligned} I_2 = &\sum_{i=1}^{m}\sum_{j<i}^{m}\sum_{\sigma,\sigma'}\prod_{k\neq i,j}^{n-1}\Big((2\pi)^3 E(\vec{p}_{k,in})\delta^3(\vec{p}_{k,in}-\vec{p}_{\sigma(k),out})\Big)\\ &\times\prod_{k'=1}^{m-2}\Big((2\pi)^3 E(\vec{k}_{k',in})\delta^3(\vec{k}_{k',in}-\vec{k}_{\sigma'(k'),out})\Big)\\ &\times iM_{2t}\left(\vec{k}_{i,in},\vec{k}_{j,in};\vec{k}_{\sigma'(i),out},\vec{k}_{\sigma'(j),out}\right)\,\mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma') \end{aligned} \tag{6.5.8}$$

$$\begin{aligned} I_3 = &\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{\sigma,\sigma'}\prod_{k\neq i,j}^{n-1}\Big((2\pi)^3 E(\vec{p}_{k,in})\delta^3(\vec{p}_{k,in}-\vec{p}_{\sigma(k),out})\Big)\\ &\times\prod_{k'=1}^{m-1}\Big((2\pi)^3 E(\vec{k}_{k',in})\delta^3(\vec{k}_{k',in}-\vec{k}_{\sigma'(k'),out})\Big)\\ &\times iM_{3}\left(\vec{p}_{i,in},\vec{k}_{j,in};\vec{p}_{\sigma(i),out},\vec{k}_{\sigma'(j),out}\right)\,\mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma') \end{aligned} \tag{6.5.9}$$

ìpou: 1) iM_1 eÐnai to anallÐwto plˆtoc skèdashc gia th diergasÐa $e^-e^- \to e^-e^-$, upologismèno se epÐpedo dèntrou, kai iM_{1t} to plˆtoc sto kanˆli t· 2) iM_2 eÐnai to anallÐwto plˆtoc skèdashc gia th diergasÐa $e^+e^+ \to e^+e^+$, upologismèno se epÐpedo dèntrou, kai iM_{2t} to plˆtoc sto kanˆli t· 3) iM_3 eÐnai to anallÐwto plˆtoc skèdashc gia th diergasÐa $e^-e^+ \to e^-e^+$, upologismèno se epÐpedo dèntrou· 4) σ (σ') metˆjesh n (m) antikeimènwn· 5) p eÐnai h orm twn hlektronÐwn kai k h orm twn pozitronÐwn. Ja sumbolÐzoume me q thn orm twn fwtonÐwn.

E^n sthn telik kat^stash up^rqei èna zeÔgoc fwtonÐwn (to opoÐo par^getai apì exaölwsh enìc zeÔgouc hlektronÐou/pozitronÐou), to pl^toc iM paÐrnei thn morf

$$\begin{aligned} iM = &\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{\sigma,\sigma'}\prod_{k\neq i}^{n-1}\Big((2\pi)^3 E(\vec{p}_{k,in})\delta^3(\vec{p}_{k,in}-\vec{p}_{\sigma(k),out})\Big) \\ &\times\prod_{k\neq j}^{m-1}\Big((2\pi)^3 E(\vec{k}_{k',in})\delta^3(\vec{k}_{k,in}-\vec{k}_{\sigma'(k'),out})\Big) \\ &\times iM_4\Big(\vec{p}_{i,in},\vec{k}_{j,in};\vec{q}_{1,out},\vec{q}_{2,out}\Big)\,\mathrm{sign}(\sigma)\,\mathrm{sign}(\sigma') \end{aligned} \quad (6.5.10)$$

ìpou iM_4 to anal loÐwto pl^toc skèdashc gia th diergasÐa $e^-e^+ \to \gamma_1\gamma_2$, upologismèno se epÐpedo dèntrou. Se t^xh e^2, den up^rqoun ^l lec peript¸seic.

Gia na upologÐsoume thn kÔria entropÐa sÔmplexhc, prèpei na upologÐsoume to tetr^gwno tou mètrou twn plat¸n iM_ϑ, kai èpeita na oloklhr¸soume wc proc tic ormèc twn telik¸n swmatidÐwn. K^je pl^toc apoteleÐ ^jroisma poll¸n ìrwn – k^je ìroc antisoiqeÐ se èna di^gramma Feynman, ìpwc exhg same prohgoumènwc. AnaptÔssontac thn èkfrash $|iM_\vartheta|^2$, prokÔptoun ìroi me ginìmena $2(m+n-2)$ sunart sewn δ (ston trisdi^stato q¸ro twn orm¸n). Se polloÔc ìrouc pou den eÐnai tèleia tetr^gwna (cross terms), oi sunart seic δ den eÐnai sumbatèc metaxÔ touc kai oi oloklhr¸seic wc proc tic telikèc ormèc dÐdoun mhdenikì apotèlesma. Up^rqoun peript¸seic cross ìrwn pou epibi¸noun, all^ h suneisfor^ eÐnai mikrìterhc t^xhc wc proc ton ìgko V , kai ètsi mporoÔme na touc amel soume sto suneqèc ìrio ìpou o ìgkoc tou koutioÔ teÐnei sto ^peiro.

H kÔria suneisfor^ proèrqetai apì ìrouc pou eÐnai tèleia tetr^gwna (thc morf c $\alpha\alpha^*$). Sthn perÐptwsh aut , oi δ-sunart seic twn dÔo ìrwn pou pollaplasi^zontai eÐnai akrib¸c oi Ðdiec. Oi δ-sunart seic tou pr¸tou ìrou anairoÔntai me ta oloklhr¸mata wc proc tic ormèc twn telik¸n swmatidÐwn kai autèc tou deÔterou ìrou dÐnoun par^gontec ìgkou. ProkÔptei ènac par^gontac ìgkou $V^{m+n-2} = V^{I-2}$ apì autèc tic $I-2$ oloklhr¸seic.

Sthn perÐptwsh pou h telik kat^stash ϑ den èqei fwtìnia, up^rqoun $n!m!/2$ tètoioi ìroi an^logoi twn tetrag¸nwn $|iM_1|^2$, $|iM_2|^2$ antÐstoiqa kai $n!m!$ ìroi an^logoi tou $|iM_3|^2$. Stouc tetragwnikoÔc ìrouc ta prìshma twn diagramm^twn den suneisfèroun.

Sth perÐptwsh pou èna zeÔgoc hlektronÐou/pozitronÐou exaul¸ntetai se 2 fwtìnia, paÐrnoume to tetr^gwno thc sqèshc 6.5.10. Kai p^li h kÔria mh mhdenik suneisfor^ proèrqetai apì ìrouc tou anaptÔgmatoc pou eÐnai tèleia tetr^gwna. Se k^je tètoio ìro up^rqoun $n+m-2$ sunart seic dèlta, oi opoÐec emplèkoun tic telikèc ormèc twn hlektronÐwn kai twn pozitronÐwn, uywmènec sto tetr^gwno. Oi oloklhr¸seic wc proc tic telikèc ormèc dÐdoun èna par^gonta ìgkou V^{I-2}. Up^rqoun $(n-1)!(m-1)!$ ìroi kai ètsi anaireÐtai o suntelest c $1/n_\vartheta!m_\vartheta!$ sthn èkfrash thc entropÐac sÔmplexhc.

Kai stic dÔo peript¸seic apomènoun dÔo oloklhr¸seic – wc proc 2 telikèc ormèc hlektronÐwn/pozitronÐwn sthn perÐptwsh thc skèdashc wc proc tic telikèc ormèc twn dÔo fwtonÐwn sthn perÐptwsh thc exaÔlwshc. Oloklhr¸nontac wc proc thn pr¸th orm epib^lletai diat rhsh thc sunolik c qwrik c orm c kai prokÔptei ènac epiprìsjetoc par^gontac ìgkou, ìpwc sthn eidik perÐptwsh twn 2 arqik¸n swmatidÐwn pou exet^same sto prohgoÔmeno kef^laio. Tèloc oloklhr¸noume wc proc to mètro thc enapomeÐnasac orm c, opìte epib^lletai diat rhsh thc sunolik c enèrgeiac kai prokÔptei o qrìnoc T tou peir^matoc.

SumbolÐzontac me θ th gwnÐa skèdashc kai me E_{ij} kai v_{ij} thn enèrgeia kai th sqetik taqÔthta twn dÔo allhlepidr¸ntwn arqik¸n swmatidÐwn sto kèntro m^zac touc, brÐskoume

$$S_{ent,sing} = \frac{T}{64\pi V} \log\left(\frac{E_d}{\lambda}\right) \int_{-1}^{1} d(\cos\theta)$$
$$\times \left\{ \sum_{i=1}^{n} \sum_{j<i}^{n} \frac{v_{ij}}{E_{ij}^2} \left|iM_1(E_{ij},\theta)\right|^2 \mathcal{B}_1(i'j';ij) + \sum_{i=1}^{m} \sum_{j<i}^{m} \frac{v_{ij}}{E_{ij}^2} \left|iM_2(E_{ij},\theta)\right|^2 \mathcal{B}_2(i'j';ij) \right.$$
$$\left. + 2\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{v_{ij}}{E_{ij}^2} \left|iM_3(E_{ij},\theta)\right|^2 \mathcal{B}_3(i'j';ij) + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{v_{ij}}{E_{ij}^2} \left|iM_4(E_{ij},\theta)\right|^2 \mathcal{B}_4(i'j';ij) \right\} \tag{6.5.11}$$

ìpou $i'j'$ ta telik^ swmatÐdia pou antistoiqoÔn se k^je mia apì tic 4 allhlepidr^seic.

Sthn perÐptwsh pou h arqik kat^stash apoteleÐtai apì n hlektrìnia paÐrnoume

$$S_{ent,sing} = \frac{T}{128\pi V} \log\left(\frac{E_d}{\lambda}\right) \sum_{ij=1}^{n} \frac{v_{ij}}{E_{ij}^2} \int_{-1}^{1} d(\cos\theta) \left|iM_1(E_{ij},\theta)\right|^2 \mathcal{B}_1(i'j';ij) \tag{6.5.12}$$

H entropÐa sÔmplexhc an^ mon^da qrìnou an^ swmatidiak puknìthta isoÔtai me

$$s_{ent,sing} = \frac{V S_{ent,sing}}{T n} = \frac{1}{128\pi\, n} \log\left(\frac{E_d}{\lambda}\right) \sum_{ij=1}^{n} \frac{v_{ij}}{E_{ij}^2} \int_{-1}^{1} d(\cos\theta) \left|iM_1(E_{ij},\theta)\right|^2 \mathcal{B}_1(i'j';ij) \tag{6.5.13}$$

kai apoklÐnei logarijmik^ sto ìrio $\lambda \to 0$.

Kef^laio 7

EntropÐa sÔmplexhc deÔterhc t^xhc

Sto prohgoÔmeno kef^laio eÐdame ìti o suntelest c thc logarijmik c apìklishc thc kÔriac entropÐac sÔmplexhc den exart^tai apì to nèfoc twn upèrujrwn fwtonÐwn pou sunodeÔoun tic asumptwtikèc katast^seic. EpÐshc diagr^mmata me eswterikoÔc brìqouc den suneisfèroun sthn kÔria t^xh thc entropÐac sÔmplexhc. Stìqoc mac se autì to kef^laio eÐnai na exet^soume thn entropÐa sÔmplexhc sth deÔterh t^xh, e^{2N+4}. Ja ergastoÔme me "gumnèc" katast^seic sth b^sh Fock, afoÔ mac endiafèrei h sumperifor^ twn upèrujrwn logarijmik¸n ìrwn, h opoÐa den exart^tai apì to ntÔsimo twn katast^sewn. jètoume loipìn $f^{\mu} = 0$. EpÐshc ja qreiasteÐ na l^boume upìyin tic katast^seic $|\beta\gamma_1\gamma_2\rangle$, afoÔ suneisfèroun sthn entropÐa sÔmplexhc se t^xh e^{2N+4}.

7.1 'Iqnh gumn¸n katast^sewn wc proc touc upèrujrouc bajmoÔc eleujerÐac

Arqik^ ja upologÐsoume ta merik^ Ðqnh di^forwn telest¸n ket-bra wc proc touc upèrujrouc bajmoÔc eleujerÐac. Mia kat^stash dÔo fwtonÐwn an kei sto upèrujro mèroc tou q¸rou Hilbert mìno e^n h enèrgeia tou k^je fwtonÐou eÐnai mikrìterh apì E kai epÐshc to ^jroisma twn energei¸n touc eÐnai mikrìtero apì E. E^n h enèrgeia tou k^je fwtonÐou eÐnai mikrìterh apì E all^ to ^jroisma twn energei¸n touc eÐnai megalÔtero apì E, h kat^stash twn dÔo fwtonÐwn an kei sto uperi¸dec mèroc tou q¸rou Hilbert.

Oi katast^seic qwrÐzontai se upèrujro kai uperi¸dec mèroc wc akoloÔjwc

$$
\begin{aligned}
&|\beta\rangle = |\beta\rangle_H \times |0\rangle_S \\
&|\beta\gamma\rangle = |\beta\rangle_H \times |\gamma\rangle_S\,, \quad \omega_\gamma < E \\
&|\beta\gamma\rangle = |\beta\gamma\rangle_H \times |0\rangle_S\,, \quad \omega_\gamma > E \\
&|\beta\gamma_1\gamma_2\rangle = |\beta\rangle_H \times |\gamma_1\gamma_2\rangle_S\,, \quad (\omega_{\gamma_1},\omega_{\gamma_2})_{\rm Soft} \\
&|\beta\gamma_1\gamma_2\rangle = |\beta\gamma_1\rangle_H \times |\gamma_2\rangle_S\,, \quad \omega_{\gamma_1} < E,\ \ \omega_{\gamma_2} > E \\
&|\beta\gamma_1\gamma_2\rangle = |\beta\gamma_1\gamma_2\rangle_H \times |0\rangle_S\,, \quad (\omega_{\gamma_1},\omega_{\gamma_2})_{\rm Hard}
\end{aligned}
\tag{7.1.1}
$$

ìpou

$$
\begin{aligned}
&|\gamma\rangle_S = \alpha^\dagger_s(\vec p)\,|0\rangle_S \\
&|\gamma_1\gamma_2\rangle_S = \alpha^\dagger_{s_1}(\vec p_1)\alpha^\dagger_{s_2}(\vec p_2)\,|0\rangle_S
\end{aligned}
\tag{7.1.2}
$$

Upenjum¸zoume ìti ta kets $|\beta\rangle$ symbolÐzoun katast^seic qwrÐc fwtìnia.

Qr simec sqèseic eÐnai oi akìloujec

$$
\begin{aligned}
&{\rm Tr}_{H_S}\,|0\rangle_S\,\langle 0|_S = 1 \\
&{\rm Tr}_{H_S}\,\big(\alpha^\dagger_s(\vec p)\,|0\rangle_S\big)\big(\,\langle 0|_S\,\big) = 0 \\
&{\rm Tr}_{H_S}\,\big(\alpha^\dagger_s(\vec p)\,|0\rangle_S\big)\big(\,\langle 0|_S\,\alpha_{s'}(\vec p\,')\big) = \delta_{ss'}\delta\vec p\,\vec p\,' \\
&{\rm Tr}_{H_S}\,\big(\alpha^\dagger_{s_1}(\vec p_1)\alpha^\dagger_{s_2}(\vec p_2)\,|0\rangle_S\big)\big(\,\langle 0|_S\,\alpha_{s'}(\vec p\,')\big) = 0 \\
&{\rm Tr}_{H_S}\,\big(\alpha^\dagger_{s_1}(\vec p_1)\alpha^\dagger_{s_2}(\vec p_2)\,|0\rangle_S\big)\big(\,\langle 0|_S\,\alpha_{s_1'}(\vec p_1\,')\alpha_{s_2'}(\vec p_2\,')\big) = \\
&\qquad\qquad \delta s_1' s_2\,\delta s_2' s_1\,\delta\vec p_1\,'\vec p_2\,\delta\vec p_2\,'\,\vec p_1 \;+\; \delta s_1' s_1\,\delta s_2' s_2\,\delta\vec p_1\,'\,\vec p_1\,\delta\vec p_2\,'\,\vec p_2
\end{aligned}
\tag{7.1.3}
$$

Arqik^ ja upologÐsoume ta Ðqnh gia peript¸seic ìpou ta fwtìnia stic katast^seic $|\beta\gamma\rangle$ kai $|\beta\gamma_1\gamma_2\rangle$ eÐnai upèrujra. Qrhsimopoi¸ntac tic pio p^nw sqèseic brÐskoume

$$
\begin{aligned}
&{\rm Tr}_{H_S}\,|\beta\rangle\,\langle\beta'| = |\beta\rangle_H\,\langle\beta'|_H \\
&{\rm Tr}_{H_S}\,|\beta\gamma\rangle\,\langle\beta'| = 0 \\
&{\rm Tr}_{H_S}\,|\beta\gamma\rangle\,\langle\beta'\gamma'| = |\beta\rangle_H\,\langle\beta'|_H\,\delta_{ss'}\delta\vec p\,\vec p\,' \\
&{\rm Tr}_{H_S}\,|\beta\gamma_1\gamma_2\rangle\,\langle\beta'| = 0 \\
&{\rm Tr}_{H_S}\,|\beta\gamma_1\gamma_2\rangle\,\langle\beta'\gamma'| = 0 \\
&{\rm Tr}_{H_S}\,|\beta\gamma_1\gamma_2\rangle\,\langle\beta'\gamma_1'\gamma_2'| = |\beta\rangle_H\,\langle\beta'|_H\,\big(\delta s_1' s_2\,\delta s_2' s_1\,\delta\vec p_1\,'\vec p_2\,\delta\vec p_2\,'\vec p_1 + \delta s_1' s_1\,\delta s_2' s_2\,\delta\vec p_1\,'\vec p_1\delta\vec p_2\,'\vec p_2\big)
\end{aligned}
\tag{7.1.4}
$$

Gia na upologÐsoume ton pÐnaka puknìthtac ja prèpei na sumperil^boume katast^seic me uperi¸dh fwtìnia. Jewr¸ntac ^jroisma wc proc ìlec tic dunatèc timèc twn

$\beta,\ \beta,\ \gamma,\ \gamma',\ \gamma_1,\ \gamma_2,\ \gamma_1',\ \gamma_2'$, prokÔptoun oi akìloujec sqèseic

$$\mathrm{Tr}_{H_S} |\beta\rangle \langle\beta'| = |\beta\rangle_H \langle\beta'|_H \tag{7.1.5}$$

$$\mathrm{Tr}_{H_S} |\beta\gamma\rangle \langle\beta'| = \sum_{\omega_\gamma > E} |\beta\gamma\rangle_H \langle\beta'|_H \tag{7.1.6}$$

$$\mathrm{Tr}_{H_S} |\beta\gamma\rangle \langle\beta'\gamma'| = \sum_{\omega_\gamma, \omega_{\gamma'} < E} |\beta\rangle_H \langle\beta'|_H \, \delta_{ss'} \, \delta\vec{p}\,\vec{p}' + \sum_{\omega_\gamma, \omega_{\gamma'} > E} |\beta\gamma\rangle_H \langle\beta'\gamma'|_H \tag{7.1.7}$$

$$\mathrm{Tr}_{H_S} |\beta\gamma_1\gamma_2\rangle \langle\beta'| = \sum_{(\gamma_1,\gamma_2)_H} |\beta\gamma_1\gamma_2\rangle_H \langle\beta'|_H \tag{7.1.8}$$

$$\mathrm{Tr}_{H_S} |\beta\gamma_1\gamma_2\rangle \langle\beta'\gamma'| = \sum_{\substack{(\gamma_1,\gamma_2)_H \\ \omega_\gamma > E}} |\beta\gamma_1\gamma_2\rangle_H \langle\beta'\gamma'|_H + \sum_{\substack{\omega_{\gamma_1}, \omega_{\gamma'} < E \\ \omega_{\gamma_2} > E}} |\beta\gamma_2\rangle_H \langle\beta'|_H \, \delta_{s_2 s'} \, \delta\vec{p}_2\,\vec{p}' \tag{7.1.9}$$

$$\mathrm{Tr}_{H_S} |\beta\gamma_1\gamma_2\rangle \langle\beta'\gamma_1'\gamma_2'| = \sum_{\substack{(\gamma_1,\gamma_2)_H \\ (\gamma_1',\gamma_2')_H}} |\beta\gamma_1\gamma_2\rangle_H \langle\beta'\gamma_1'\gamma_2'|_H$$
$$+ \sum_{\substack{\omega_{\gamma_2}, \omega_{\gamma_2'} > E \\ \omega_{\gamma_1} < E}} |\beta\gamma_2\rangle_H \langle\beta'\gamma_2'|_H \, \delta s_1' s_1 \delta\vec{q}_1'\vec{q}_1 + \sum_{(\gamma_1,\gamma_2)_S} |\beta\rangle_H \langle\beta'|_H \, \delta s_1' s_1 \delta s_2' s_2 \delta\vec{q}_1'\vec{q}_1 \delta\vec{q}_2'\vec{q}_2 \tag{7.1.10}$$

7.2 Pðnakac puknìthtac

'Opwc kai sto prohgoÔmeno kefˆlaio grˆfoume thn telik katˆstash sthn akìloujh morf , qrhsimopoi¸ntac mia pl rh bˆsh

$$|\Psi\rangle_{out} = |\alpha\rangle + \sum_\beta A_{\beta\alpha} |\beta\rangle + \sum_{\beta,\gamma} B_{\beta\gamma,\alpha} |\beta\gamma\rangle + \sum_{\substack{\beta,\gamma_1,\gamma_2 \\ \omega_{\gamma_1} \le \omega_{\gamma_2}}} \Gamma_{\beta\gamma_1\gamma_2,\alpha} |\beta\gamma_1\gamma_2\rangle + \dots \tag{7.2.1}$$

Upenjumðzoume ìti ta pl^th $A_{\beta\alpha}$, $B_{\beta\gamma,\alpha}$ kai $\Gamma_{\beta\gamma_1\gamma_2,\alpha}$ eðnai thc t^xhc e^N, e^{N+1} kai e^{N+2} antðstoiqa. O pðnakac puknìthtac twn uperiwd¸n bajm¸n eleujerðac gr^fetai sthn morf

$$
\begin{aligned}
\rho_H = \mathrm{Tr}_{H_S} |\Psi\rangle_{out} \langle\Psi|_{out} = \\
|\alpha\rangle_H \langle\alpha|_H + \left(A_{\alpha\beta} |\beta\rangle_H + \sum_{\omega_\gamma > E} B_{\beta\gamma,\alpha} |\beta\gamma\rangle_H + \sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha} |\beta\gamma_1\gamma_2\rangle_H + \dots \right) \langle\alpha|_H \\
+ |\alpha\rangle_H \left(A^*_{\alpha\beta'} \langle\beta'|_H + \sum_{\omega_{\gamma'} > E} B^*_{\beta'\gamma',\alpha} \langle\beta'\gamma'|_H + \sum_{(\gamma'_1,\gamma'_2)_H} \Gamma^*_{\beta'\gamma'_1\gamma'_2,\alpha} \langle\beta'\gamma'_1\gamma'_2|_H + \dots \right) \\
+ D_{\beta,\beta'} |\beta\rangle_H \langle\beta'|_H + \sum_{\omega_{\gamma'} > E} D_{\beta,\beta'\gamma'} |\beta\rangle_H \langle\beta'\gamma'|_H + \sum_{\omega_\gamma > E} D_{\beta\gamma,\beta'} |\beta\gamma\rangle_H \langle\beta'|_H \\
+ \sum_{\omega_\gamma,\omega_{\gamma'} > E} D_{\beta\gamma,\beta'\gamma'} |\beta\gamma\rangle_H \langle\beta'\gamma'|_H \\
+ \sum_{(\gamma'_1,\gamma'_2)_H} A_{\beta\alpha} \Gamma^*_{\beta'\gamma'_1\gamma'_2,\alpha} |\beta\rangle_H \langle\beta'\gamma'_1\gamma'_2|_H + \sum_{(\gamma_1,\gamma_2)_H} A^*_{\beta'\alpha} \Gamma_{\beta\gamma_1\gamma_2,\alpha} |\beta\gamma_1\gamma_2\rangle_H \langle\beta'|_H + \dots \\
+ \sum_{\substack{(\gamma'_1,\gamma'_2)_H \\ \omega_\gamma > E}} B_{\beta\gamma,\alpha} \Gamma^*_{\beta'\gamma'_1\gamma'_2,\alpha} |\beta\gamma\rangle_H \langle\beta'\gamma'_1\gamma'_2|_H + \sum_{\substack{(\gamma_1,\gamma_2)_H \\ \omega'_\gamma > E}} B^*_{\beta'\gamma',\alpha} \Gamma_{\beta\gamma_1\gamma_2,\alpha} |\beta\gamma_1\gamma_2\rangle_H \langle\beta'\gamma'|_H + \dots \\
+ \sum_{\substack{(\gamma_1,\gamma_2)_H \\ (\gamma'_1,\gamma'_2)_H}} \Gamma^*_{\beta'\gamma'_1\gamma'_2,\alpha} \Gamma_{\beta\gamma_1\gamma_2,\alpha} |\beta\gamma_1\gamma_2\rangle_H \langle\beta'\gamma'_1\gamma'_2|_H + \dots
\end{aligned}
\tag{7.2.2}
$$

ìpou

$$D_{\beta,\beta'} = A_{\beta\alpha} A^*_{\beta'\alpha} + \sum_{\omega_\gamma < E} B_{\beta\gamma,\alpha} B^*_{\beta'\gamma,\alpha} + \sum_{(\gamma_1,\gamma_2)_S} \Gamma^*_{\beta'\gamma_1\gamma_2,\alpha} \Gamma_{\beta\gamma_1\gamma_2,\alpha} + \dots \tag{7.2.3}$$

$$D_{\beta\gamma,\beta'} = A^*_{\beta'\alpha} B_{\beta\gamma,\alpha} + \sum_{\omega_{\gamma'} < E} B^*_{\beta'\gamma',\alpha} \Gamma_{\beta\gamma\gamma',\alpha} + \dots \tag{7.2.4}$$

$$D_{\beta,\beta'\gamma'} = A_{\beta\alpha} B^*_{\beta'\gamma',\alpha} + \sum_{\omega_\gamma < E} B_{\beta\gamma,\alpha} \Gamma^*_{\beta'\gamma\gamma',\alpha} + \dots \tag{7.2.5}$$

$$D_{\beta\gamma,\beta'\gamma'} = B_{\beta\gamma,\alpha} B^*_{\beta'\gamma',\alpha} + \sum_{\omega_{\gamma_1} < E} \Gamma_{\beta\gamma\gamma_1,\alpha} \Gamma^*_{\beta'\gamma'\gamma_1,\alpha} + \dots \tag{7.2.6}$$

Ajroðzoume wc proc ìlec tic timèc twn katast^sewn $|\beta\rangle$, $|\beta'\rangle$. Oi teleðec stic tèsseric teleutaðec sqèseic perilamb^noun ìrouc megalÔterhc t^xhc apì e^{2N+4}, ìpwc kai oi teleðec sthn teleutaða gramm thc sqèshc 7.2.2. Oi teleðec stic upìloipec grammèc thc sqèshc 7.2.2, perilamb^noun katast^seic me perissìtera apì dÔo fwtìnia. Oi ìroi autoð den suneisfèroun sthn entropða sÔmplexhc se deÔterh t^xh.

Gia na mporèsoume na upologðsoume ta di^fora ðqnh ja qreiastoÔme ta eswterik^

ginìmena metaxÔ twn katast^sewn pou emfanðzontai ston pðnaka puknìthtac. Jewr¸ntac ìti h arqik kat^stash den perilamb^nei fwtìnia, paðrnoume tic akìloujec sqèseic

$$
\begin{aligned}
&\langle\alpha|\alpha\rangle = 1\\
&\langle\alpha|\beta\rangle = \delta_{\alpha\beta}\\
&\langle\alpha|\beta\gamma\rangle = 0\\
&\langle\alpha|\beta\gamma_1\gamma_2\rangle = 0\\
&\left\langle\beta'\middle|\beta\right\rangle = \delta_{\beta'\beta}\\
&\left\langle\beta'\middle|\beta\gamma\right\rangle = 0\\
&\left\langle\beta'\middle|\beta\gamma_1\gamma_2\right\rangle = 0\\
&\left\langle\beta'\gamma'\middle|\beta\gamma\right\rangle = \delta_{\beta'\beta}\delta_{\gamma'\gamma}\\
&\left\langle\beta'\gamma'\middle|\beta\gamma_1\gamma_2\right\rangle = 0\\
&\left\langle\beta'\gamma_1'\gamma_2'\middle|\beta\gamma_1\gamma_2\right\rangle = \delta_{\beta'\beta}\delta_{(\gamma_1'\gamma_2')(\gamma_1\gamma_2)}
\end{aligned}
\tag{7.2.7}
$$

Gr^fontac thc telik kat^stash sthn morf

$$|\Psi\rangle_{out} = S\,|\alpha\rangle \tag{7.2.8}$$

kai qrhsimopoi¸ntac thn sqèsh monadiakìthtac, $S^{\dagger}S = 1$, brðskoume ìti

$$0 = A_{\alpha\alpha} + A^*_{\alpha\alpha} + \sum_{\beta} A_{\beta\alpha}A^*_{\beta\alpha} + \sum_{\beta,\gamma} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + \sum_{\substack{\beta,\gamma_1,\gamma_2\\ \omega_{\gamma_1}\leq\omega_{\gamma_2}}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} + \ldots \tag{7.2.9}$$

'Ara loipìn

$$A_{\alpha\alpha} + A^*_{\alpha\alpha} = \sum_{\beta} A_{\beta\alpha}A^*_{\beta\alpha} + \sum_{\beta,\gamma} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + \sum_{\substack{\beta,\gamma_1,\gamma_2\\ \omega_{\gamma_1}\leq\omega_{\gamma_2}}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} + \ldots \tag{7.2.10}$$

eðnai thc t^xhc e^{2N} kai

$$A_{\alpha\alpha} + A^*_{\alpha\alpha} + \sum_{\beta} A_{\beta\alpha}A^*_{\beta\alpha} = \sum_{\beta,\gamma} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + \sum_{\substack{\beta,\gamma_1,\gamma_2\\ \omega_{\gamma_1}\leq\omega_{\gamma_2}}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} + \ldots \tag{7.2.11}$$

thc t^xhc e^{2N+2}.

7.3 EntropÐa sÔmplexhc

UpenjumÐzoume ìti oi entropÐec Renyi dÐnontai apì thn tÔpo

$$S_m = \frac{1}{1-m} \log \mathrm{Tr}(\rho_H)^m \tag{7.3.1}$$

Gr^fontac ton pÐnaka puknìthtac sthn morf $\rho_H = |\alpha\rangle \langle\alpha| + \epsilon \equiv \rho_0 + \epsilon$, o pr¸toc ìroc eÐnai t^xhc 1 kai o deÔteroc ìroc eÐnai thc t^xhc e^N. Sthn perÐptwsh ìpou $N = 2$, ja prèpei na krat soume ìrouc t^xhc ϵ^4, ¸ste na upologÐsoume thn entropÐa sÔmplexhc se t^xh e^{2N+4}.

Qrhsimopoi¸ntac th sqèsh $\rho_0^2 = \rho_0$ kai thn kuklik idiìthta tou Ðqnouc, arkeÐ na upologÐsoume ta Ðqnh twn posot twn $\epsilon, \rho_0\epsilon$, ϵ^2, $\rho_0\epsilon^2$, $\rho_0\epsilon\rho_0\epsilon$, ϵ^3, $\rho_0\epsilon^3$, $\rho_0\epsilon^2\rho_0\epsilon$, $(\rho_0\epsilon)^3$, ϵ^4, $\rho_0\epsilon^4, \rho_0\epsilon^3\rho_0\epsilon$, $\rho_0\epsilon^2\rho_0\epsilon^2, \rho_0\epsilon^2(\rho_0\epsilon)^2, (\rho_0\epsilon)^4$.

7.3.1 'Iqnh wc proc upèruj rouc bajmoÔc eleujerÐac

Arqik^ upenjumÐzoume ìti

$$\mathrm{Tr}\, |A\rangle \langle B| \equiv \langle B|A\rangle \tag{7.3.2}$$

Gia ta Ðqnh t^xhc ϵ paÐrnoume

$$\mathrm{Tr}\, \epsilon = 0 \tag{7.3.3}$$

(afoÔ $\mathrm{Tr}\, \rho_H = \mathrm{Tr}\, \rho_0 = 1$) kai

$$\begin{aligned} \mathrm{Tr}\, \rho_0\epsilon &= A_{\alpha\alpha} + A^*_{\alpha\alpha} + D_{\alpha,\alpha} \\ &= A_{\alpha\alpha} + A^*_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha} + \sum_{\omega\gamma<E} B_{\alpha\gamma,\alpha}B^*_{\alpha\gamma,\alpha} + \sum_{(\gamma_1,\gamma_2)_S} \Gamma_{\alpha\gamma_1\gamma_2,\alpha}\Gamma^*_{\alpha\gamma_1\gamma_2,\alpha} \end{aligned} \tag{7.3.4}$$

Krat same ìrouc t^xhc e^{2N+4}.

Gia ta Ðqnh t^xhc ϵ^2, brÐskoume

$$\mathrm{Tr}\,\epsilon^2 = A_{\alpha\alpha} + A^*_{\alpha\alpha} + D_{\alpha,\alpha}$$

$$= A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + 2A_{\beta\alpha}A^*_{\beta\alpha} + 2\sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + 2\sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha}$$
$$+2A_{\beta\alpha}D_{\alpha,\beta} + 2A^*_{\beta\alpha}D_{\beta\alpha} + 2\sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}D_{\alpha,\beta\gamma} + 2\sum_{\omega\gamma>E} B^*_{\beta\gamma,\alpha}D_{\beta\gamma,\alpha} + D_{\beta,\beta'}D_{\beta',\beta}$$

$$= A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + 2A_{\beta\alpha}A^*_{\beta\alpha} + 2\sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + 2\sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha}$$
$$+2\sum_{\omega\gamma<E}\Big(A_{\beta\alpha}B_{\alpha\gamma,\alpha}B^*_{\beta\gamma,\alpha} + A^*_{\beta\alpha}B_{\beta\gamma,\alpha}B^*_{\alpha\gamma,\alpha}\Big) \tag{7.3.5}$$

Sto teleutaÐo b ma qrhsimopoi same tic sqèseic 7.2.10 kai 7.2.11. Me an^logo trìpo brÐskoume

$$\mathrm{Tr}\,\rho_0\epsilon^2 = A^2_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha}$$
$$+(A_{\alpha\alpha} + A^*_{\alpha\alpha})D_{\alpha,\alpha} + A_{\beta\alpha}D_{\alpha,\beta} + A^*_{\beta\alpha}D_{\beta,\alpha} + \sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha}$$
$$+\sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}D_{\alpha,\beta\gamma} + \sum_{\omega\gamma>E} B^*_{\beta\gamma,\alpha}D_{\beta\gamma,\alpha} + \sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} + D_{\alpha,\beta}D_{\beta,\alpha}$$

$$= A^2_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha}(1 + A_{\alpha\alpha} + A^*_{\alpha\alpha}) + \sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha}$$
$$+\sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} + \sum_{\omega\gamma<E}\Big(A_{\beta\alpha}B_{\alpha\gamma,\alpha}B^*_{\beta\gamma,\alpha} + A^*_{\beta\alpha}B_{\beta\gamma,\alpha}B^*_{\alpha\gamma,\alpha}\Big) \tag{7.3.6}$$

kai

$$\mathrm{Tr}\,\rho_0\epsilon\rho_0\epsilon = A^2_{\alpha\alpha} + 2A_{\alpha\alpha}A^*_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + 2(A_{\alpha\alpha} + A^*_{\alpha\alpha})D_{\alpha,\alpha} + D^2_{\alpha,\alpha}$$
$$= (A_{\alpha\alpha} + A^*_{\alpha\alpha})^3 + (A_{\alpha\alpha}A^*_{\alpha\alpha})^2 \tag{7.3.7}$$

Gia touc ìrouc t^xhc ϵ^3, brÐskoume

$$\mathrm{Tr}\,\epsilon^3 = A^3_{\alpha\alpha} + A^{*3}_{\alpha\alpha} + 3(A_{\alpha\alpha} + A^*_{\alpha\alpha} + A_{\beta'\alpha}A^*_{\beta'\alpha} + A^2_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha} + A^{*2}_{\alpha\alpha})A_{\beta\alpha}A^*_{\beta\alpha}$$
$$+3\sum_{\omega\gamma>E}(A_{\alpha\alpha} + A^*_{\alpha\alpha})B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha}$$
$$= A^3_{\alpha\alpha} + A^{*3}_{\alpha\alpha} + 3A_{\beta\alpha}A^*_{\beta\alpha}(A^2_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha} + A^{*2}_{\alpha\alpha}) \tag{7.3.8}$$

$$\begin{aligned}
\operatorname{Tr}\rho_0\epsilon^3 = A_{\alpha\alpha}^3 + A_{\alpha\alpha}^{*3} + A_{\alpha\alpha}^2 A_{\alpha\alpha}^* + A_{\alpha\alpha}A_{\alpha\alpha}^{*2} + A_{\alpha\alpha}^3 A_{\alpha\alpha}^* + A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} + A_{\alpha\alpha}A_{\alpha\alpha}^{*3} \\
+A_{\beta\alpha}A_{\beta\alpha}^*(2A_{\alpha\alpha} + 2A_{\alpha\alpha}^* + 4A_{\alpha\alpha}A_{\alpha\alpha}^* + 2A_{\alpha\alpha}^2 + 2A_{\alpha\alpha}^{*2} + A_{\beta'\alpha}A_{\beta'\alpha}^*) \\
+2\sum_{\omega\gamma>E}(A_{\alpha\alpha} + A_{\alpha\alpha}^*)B_{\beta\gamma,\alpha}B_{\beta\gamma,\alpha}^* \\
= (A_{\alpha\alpha} + A_{\alpha\alpha}^* + A_{\alpha\alpha}A_{\alpha\alpha}^*)(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2}) + A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} + (A_{\alpha\alpha} + A_{\alpha\alpha}^*)A_{\beta\alpha}A_{\beta\alpha}^*
\end{aligned} \tag{7.3.9}$$

$$\begin{aligned}
\operatorname{Tr}\rho_0\epsilon^2\rho_0\epsilon = A_{\alpha\alpha}^3 + A_{\alpha\alpha}^{*3} + 2A_{\alpha\alpha}^2 A_{\alpha\alpha}^* + 2A_{\alpha\alpha}A_{\alpha\alpha}^{*2} + 3A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} + 2A_{\alpha\alpha}^3 A_{\alpha\alpha}^* + 2A_{\alpha\alpha}A_{\alpha\alpha}^{*3} \\
+A_{\beta\alpha}A_{\beta\alpha}^*(A_{\alpha\alpha} + A_{\alpha\alpha}^* + A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + 4A_{\alpha\alpha}A_{\alpha\alpha}^*) + \sum_{\omega\gamma>E}(A_{\alpha\alpha} + A_{\alpha\alpha}^*)B_{\beta\gamma,\alpha}B_{\beta\gamma,\alpha}^* \\
= 3A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} + A_{\beta\alpha}A_{\beta\alpha}^*(A_{\alpha\alpha} + A_{\alpha\alpha}^* + 3A_{\alpha\alpha}A_{\alpha\alpha}^*)
\end{aligned} \tag{7.3.10}$$

Tèloc

$$\begin{aligned}
\operatorname{Tr}(\rho_0\epsilon)^3 = A_{\alpha\alpha}^3 + A_{\alpha\alpha}^{*3} + 3A_{\alpha\alpha}A_{\alpha\alpha}^*(A_{\alpha\alpha} + A_{\alpha\alpha}^* + A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + 2A_{\alpha\alpha}A_{\alpha\alpha}^*) \\
= (A_{\alpha\alpha} + A_{\alpha\alpha}^*)^3 + 3A_{\alpha\alpha}A_{\alpha\alpha}^*(A_{\alpha\alpha} + A_{\alpha\alpha}^*)^2 = 0
\end{aligned} \tag{7.3.11}$$

to opoðo den suneisfèrei se t^xh e^{2N+4}.

Oi ìroi t^xhc ϵ^4 eðnai

$$\begin{aligned}
\operatorname{Tr}\epsilon^4 = A_{\alpha\alpha}^4 + A_{\alpha\alpha}^{*4} + 2A_{\beta\alpha}A_{\beta\alpha}^*(2A_{\alpha\alpha}^2 + 2A_{\alpha\alpha}^{*2} + 2A_{\alpha\alpha}A_{\alpha\alpha}^* + A_{\beta'\alpha}A_{\beta'\alpha}^*) \\
= A_{\alpha\alpha}^4 + A_{\alpha\alpha}^{*4} + 2A_{\beta\alpha}A_{\beta\alpha}^*(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + A_{\beta'\alpha}A_{\beta'\alpha}^*)
\end{aligned} \tag{7.3.12}$$

$$\begin{aligned}
\operatorname{Tr}\rho_0\epsilon^4 = A_{\alpha\alpha}^4 + A_{\alpha\alpha}^{*4} + A_{\alpha\alpha}^3 A_{\alpha\alpha}^* + A_{\alpha\alpha}A_{\alpha\alpha}^{*3} + A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} \\
+A_{\beta\alpha}A_{\beta\alpha}^*(3A_{\alpha\alpha}^2 + 3A_{\alpha\alpha}^{*2} + 4A_{\alpha\alpha}A_{\alpha\alpha}^* + A_{\beta'\alpha}A_{\beta'\alpha}^*) \\
= A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} + A_{\beta\alpha}A_{\beta\alpha}^*(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + A_{\beta'\alpha}A_{\beta'\alpha}^*)
\end{aligned} \tag{7.3.13}$$

$$\begin{aligned}
\operatorname{Tr}\rho_0\epsilon^2\rho_0\epsilon^2 = A_{\alpha\alpha}^4 + A_{\alpha\alpha}^{*4} + 2A_{\alpha\alpha}^3 A_{\alpha\alpha}^* + 2A_{\alpha\alpha}A_{\alpha\alpha}^{*3} + 3A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} \\
+A_{\beta\alpha}A_{\beta\alpha}^*(2A_{\alpha\alpha}^2 + 2A_{\alpha\alpha}^{*2} + 2A_{\alpha\alpha}A_{\alpha\alpha}^* + A_{\beta'\alpha}A_{\beta'\alpha}^*) \\
= A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} + A_{\beta\alpha}A_{\beta\alpha}^*(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + A_{\beta'\alpha}A_{\beta'\alpha}^*)
\end{aligned} \tag{7.3.14}$$

$$\begin{aligned}
\operatorname{Tr}\rho_0\epsilon^3\rho_0\epsilon = A_{\alpha\alpha}^4 + A_{\alpha\alpha}^{*4} + 2A_{\alpha\alpha}^3 A_{\alpha\alpha}^* + 2A_{\alpha\alpha}A_{\alpha\alpha}^{*3} + 2A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} \\
+2A_{\beta\alpha}A_{\beta\alpha}^*(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + 2A_{\alpha\alpha}A_{\alpha\alpha}^*) \\
= (A_{\alpha\alpha} + A_{\alpha\alpha}^*)^2(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + A_{\beta\alpha}A_{\beta\alpha}^*) = 0
\end{aligned} \tag{7.3.15}$$

$$\begin{aligned}\operatorname{Tr}\rho_0\epsilon^2(\rho_0\epsilon)^2 &= A_{\alpha\alpha}^4 + A_{\alpha\alpha}^{*4} + 3A_{\alpha\alpha}^3 A_{\alpha\alpha}^* + 3A_{\alpha\alpha}A_{\alpha\alpha}^{*3} + 4A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} \\ &\quad + A_{\beta\alpha}A_{\beta\alpha}^*(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + 2A_{\alpha\alpha}A_{\alpha\alpha}^*) \\ &= (A_{\alpha\alpha} + A_{\alpha\alpha}^*)^2(A_{\alpha\alpha}^2 + A_{\alpha\alpha}^{*2} + A_{\alpha\alpha}A_{\alpha\alpha}^* + A_{\beta\alpha}A_{\beta\alpha}^*) = 0 \end{aligned} \tag{7.3.16}$$

$$\begin{aligned}\operatorname{Tr}(\rho_0\epsilon)^4 &= A_{\alpha\alpha}^4 + A_{\alpha\alpha}^{*4} + 4A_{\alpha\alpha}^3 A_{\alpha\alpha}^* + 4A_{\alpha\alpha}A_{\alpha\alpha}^{*3} + 6A_{\alpha\alpha}^2 A_{\alpha\alpha}^{*2} \\ &= (A_{\alpha\alpha} + A_{\alpha\alpha}^*)^4 = 0 \end{aligned} \tag{7.3.17}$$

Qrhsimopoi¸ntac thn sqèsh 7.2.10, mporoÔme na antikatast soume touc parˆgontec $A_{\beta\alpha}A_{\beta\alpha}^*$ me $-(A_{\alpha\alpha} + A_{\alpha\alpha}^*)$ se ìla ta ðqnh tˆxhc ϵ^3 kai ϵ^4.

7.3.2 Anˆptugma dunˆmewn tou pðnaka puknìthtac

Gia na broÔme tic entropðec Renyi prèpei na anaptÔxoume to ðqnoc tou pðnaka $(\rho_H)^m$ sunart sei twn iqn¸n pou upologðsame sto prohgoÔmeno kefˆlaio. Upenjumðzoume ìti

$$\operatorname{Tr}\rho_H = \operatorname{Tr}\rho_0 + \operatorname{Tr}\epsilon = 1 \tag{7.3.18}$$

kai

$$\operatorname{Tr}(\rho_H)^2 = \operatorname{Tr}(\rho_0)^2 + 2\operatorname{Tr}\rho_0\epsilon + \operatorname{Tr}\epsilon^2 = \operatorname{Tr}(\rho_0) + 2\operatorname{Tr}\rho_0\epsilon + \operatorname{Tr}\epsilon^2 \tag{7.3.19}$$

AkoloÔjwc paðrnoume

$$\operatorname{Tr}(\rho_H)^3 = \operatorname{Tr}(\rho_0) + 3\operatorname{Tr}\rho_0\epsilon + 3\operatorname{Tr}\rho_0\epsilon^2 + \operatorname{Tr}\epsilon^3 \tag{7.3.20}$$

$$\operatorname{Tr}(\rho_H)^4 = \operatorname{Tr}(\rho_0) + 4\operatorname{Tr}\rho_0\epsilon + 4\operatorname{Tr}\rho_0\epsilon^2 + 2\operatorname{Tr}\rho_0\epsilon\rho_0\epsilon + 4\operatorname{Tr}\rho_0\epsilon^3 + \operatorname{Tr}\epsilon^4 \tag{7.3.21}$$

Gia $m > 4$, brðskoume [1]

$$\begin{aligned}\operatorname{Tr}(\rho_H)^m &= \operatorname{Tr}\rho_0 + m\operatorname{Tr}\rho_0\epsilon + m\operatorname{Tr}\rho_0\epsilon^2 + \frac{m(m-3)}{2}\operatorname{Tr}\rho_0\epsilon\rho_0\epsilon \\ &\quad + m\operatorname{Tr}\rho_0\epsilon^3 + m(m-4)\operatorname{Tr}\rho_0\epsilon^2\rho_0\epsilon + \frac{m(m-4)(m-5)}{6}\operatorname{Tr}(\rho_0\epsilon)^3 \\ &\quad + m\operatorname{Tr}\rho_0\epsilon^4 + m(m-5)\operatorname{Tr}\rho_0\epsilon^3\rho_0\epsilon + \frac{m(m-5)}{2}\operatorname{Tr}\rho_0\epsilon^2\rho_0\epsilon^2 \\ &\quad + \frac{(m-1)(m-5)(m-6)}{2}\operatorname{Tr}\rho_0\epsilon^2(\rho_0\epsilon)^2 + P(m)\operatorname{Tr}(\rho_0\epsilon)^4 \end{aligned} \tag{7.3.22}$$

[1] Οι συντελεστές υπολογίστηκαν σε επόμενο κεφάλαιο.

ìpou o suntelest c $P(m)$ dÐnetai apì thn sqèsh

$$P(m) = \frac{m(m-1)(m-2)(m-3)}{4!} - \left(m + \frac{3m(m-5)}{2} + \frac{(m-1)(m-5)(m-6)}{2}\right) \tag{7.3.23}$$

'Oroi t^xhc ϵ^5 paraleÐpontai.

Qrhsimopoi¸ntac ti sqèseic 7.3.3-7.3.17 mporoÔme na paraleÐyoume tou ìrouc pou den suneisfèroun se t^xh e^{2N+4}. PaÐrnoume loipìn

$$\begin{aligned}\mathrm{Tr}(\rho_H)^m &= \mathrm{Tr}\,\rho_0 + m\,\mathrm{Tr}\,\rho_0\epsilon + m\,\mathrm{Tr}\,\rho_0\epsilon^2 + \frac{m(m-3)}{2}\,\mathrm{Tr}\,\rho_0\epsilon\rho_0\epsilon + m\,\mathrm{Tr}\,\rho_0\epsilon^3\\ &+m(m-4)\,\mathrm{Tr}\,\rho_0\epsilon^2\rho_0\epsilon + m\,\mathrm{Tr}\,\rho_0\epsilon^4 + \frac{m(m-5)}{2}\,\mathrm{Tr}\,\rho_0\epsilon^2\rho_0\epsilon^2\end{aligned} \tag{7.3.24}$$

7.3.3 EntropÐec Renyi

Gia $m = 2$, brÐskoume

$$\mathrm{Tr}(\rho_H)^2 = \mathrm{Tr}\,\rho_0 + 2\,\mathrm{Tr}\,\rho_0\epsilon + \mathrm{Tr}\,\epsilon^2$$

$$\begin{aligned}&= 1 + 2(A_{\alpha\alpha} + A^*_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha}) + (A_{\alpha\alpha} + A^{*2}_{\alpha\alpha})^2\\ &+2\sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + 2\sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha}\\ &+2\sum_{\omega\gamma<E}(B_{\alpha\gamma,\alpha}B^*_{\alpha\gamma,\alpha} + A_{\beta\alpha}B_{\alpha\gamma,\alpha}B^*_{\beta\gamma,\alpha} + A^*_{\beta\alpha}B_{\beta\gamma,\alpha}B^*_{\alpha\gamma,\alpha}) + 2\sum_{(\gamma_1,\gamma_2)_S}\Gamma_{\alpha\gamma_1\gamma_2,\alpha}\Gamma^*_{\alpha\gamma_1\gamma_2,\alpha}\end{aligned}$$

$$\begin{aligned}&= 1 - 2\sum_{\omega\gamma<E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} - 2\sum_{\substack{\gamma_1,\gamma_2\\-(\gamma_1,\gamma_2)_H}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} + (A_{\alpha\alpha} + A^*_{\alpha\alpha})^2\\ &+2\sum_{\omega\gamma<E}(B_{\alpha\gamma,\alpha}B^*_{\alpha\gamma,\alpha} + A_{\beta\alpha}B_{\alpha\gamma,\alpha}B^*_{\beta\gamma,\alpha} + A^*_{\beta\alpha}B_{\beta\gamma,\alpha}B^*_{\alpha\gamma,\alpha}) + 2\sum_{(\gamma_1,\gamma_2)_S}\Gamma_{\alpha\gamma_1\gamma_2,\alpha}\Gamma^*_{\alpha\gamma_1\gamma_2,\alpha}\end{aligned} \tag{7.3.25}$$

Sto teleutaÐo b ma qrhsimopoi same thn sqèsh monadiakìthtac. H teleutaÐa gramm thc pio p^nw sqèshc eÐnai mhdèn exaitÐac thc diat rhshc thc enèrgeiac. To ^j roisma ston trÐto ìro perilamb^nei peript¸seic ìpou to èna fwtìnio eÐnai upèrujro kai to ^llo uperi¸dec. H enèrgeia tou uperi¸douc fwtonÐou den mporeÐ na eÐnai megalÔterh apì thn enèrgeia thc arqik c kat^stashc. 'Etsi den up^rqei k^poioc uperi¸dhc apeirismìc.

Gia $m = 3$ kai $m = 4$ paðrnoume

$$\mathrm{Tr}(\rho_H)^3 = \mathrm{Tr}\,\rho_0 + 3\,\mathrm{Tr}\,\rho_0\epsilon + 3\,\mathrm{Tr}\,\rho_0\epsilon^2 + \mathrm{Tr}\,\epsilon^3$$

$$\begin{aligned}
&= 1 + 3(A_{\alpha\alpha} + A^*_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha}) + 3(A_{\alpha\alpha} + A^*_{\alpha\alpha})^2 + 3A_{\beta\alpha}A^*_{\beta\alpha}(A_{\alpha\alpha} + A^*_{\alpha\alpha}) \\
&\quad +3\sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + 3\sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} \\
&\quad +A^3_{\alpha\alpha} + A^{*3}_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha}(A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha})
\end{aligned}$$

$$\begin{aligned}
&= 1 - 3\sum_{\omega\gamma<E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} - 3\sum_{\substack{\gamma_1,\gamma_2\\-(\gamma_1,\gamma_2)_H}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} + 3(A_{\alpha\alpha} + A^*_{\alpha\alpha})^2 \\
&\quad +A^3_{\alpha\alpha} + A^{*3}_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha}(A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha})
\end{aligned} \tag{7.3.26}$$

$$\mathrm{Tr}(\rho_H)^4 = \mathrm{Tr}\,\rho_0 + 4\,\mathrm{Tr}\,\rho_0\epsilon + 4\,\mathrm{Tr}\,\rho_0\epsilon^2 + 2\,\mathrm{Tr}\,\rho_0\epsilon\rho_0\epsilon + 4\,\mathrm{Tr}\,\rho_0\epsilon^3 + \mathrm{Tr}\,\epsilon^4$$

$$\begin{aligned}
&= 1 + 4\left(A_{\alpha\alpha} + A^*_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha} + \sum_{\omega\gamma>E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + \sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha}\right) \\
&\quad +2(A_{\alpha\alpha} + A^*_{\alpha\alpha})(3A_{\alpha\alpha} + 3A^*_{\alpha\alpha} + 2A^2_{\alpha\alpha} + 2A^{*2}_{\alpha\alpha} + 2A_{\alpha\alpha}A^*_{\alpha\alpha}) \\
&\quad +4A_{\alpha\alpha}A^*_{\alpha\alpha}(A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha} + A_{\alpha\alpha}A^*_{\alpha\alpha}) + A^4_{\alpha\alpha} + A^{*4}_{\alpha\alpha} + 2A_{\beta\alpha}A^*_{\beta\alpha}A_{\beta'\alpha}A^*_{\beta'\alpha}
\end{aligned}$$

$$\begin{aligned}
&= 1 - 4\sum_{\omega\gamma<E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} - 4\sum_{\substack{\gamma_1,\gamma_2\\-(\gamma_1,\gamma_2)_H}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} \\
&\quad +2(A_{\alpha\alpha} + A^*_{\alpha\alpha})(A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha}) - 2A^2_{\alpha\alpha}A^{*2}_{\alpha\alpha} + 2A_{\beta\alpha}A^*_{\beta\alpha}A_{\beta'\alpha}A^*_{\beta'\alpha}
\end{aligned} \tag{7.3.27}$$

Tèloc gia $m > 4$ br□skoume

$$\mathrm{Tr}(\rho_H)^m = \mathrm{Tr}\,\rho_0 + m\,\mathrm{Tr}\,\rho_0\epsilon + m\,\mathrm{Tr}\,\rho_0\epsilon^2 + \frac{m(m-3)}{2}\,\mathrm{Tr}\,\rho_0\epsilon\rho_0\epsilon + m\,\mathrm{Tr}\,\rho_0\epsilon^3$$
$$+m(m-4)\,\mathrm{Tr}\,\rho_0\epsilon^2\rho_0\epsilon + m\,\mathrm{Tr}\,\rho_0\epsilon^4 + \frac{m(m-5)}{2}\,\mathrm{Tr}\,\rho_0\epsilon^2\rho_0\epsilon^2$$

$$= 1 + m\Big(A_{\alpha\alpha} + A^*_{\alpha\alpha} + A_{\beta\alpha}A^*_{\beta\alpha} + \sum_{\omega\gamma > E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + \sum_{(\gamma_1,\gamma_2)_H} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha}\Big)$$
$$+(A_{\alpha\alpha} + A^*_{\alpha\alpha})\Big[(A_{\alpha\alpha} + A^*_{\alpha\alpha})\frac{m(m-1)}{2} + m(m-1)A_{\beta\alpha}A^*_{\beta\alpha} + m(m-3)A_{\alpha\alpha}A^*_{\alpha\alpha}$$
$$+m(A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha})\Big] + mA_{\alpha\alpha}A^*_{\alpha\alpha}(A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha}) + \frac{m(7m-25)}{2}A^2_{\alpha\alpha}A^{*2}_{\alpha\alpha}$$
$$+A_{\beta\alpha}A^*_{\beta\alpha}\Big[\frac{m(m-5)}{2}(A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha}) + 3m(m-4)A_{\alpha\alpha}A^*_{\alpha\alpha} + \frac{m(m-3)}{2}A_{\beta'\alpha}A^*_{\beta'\alpha}\Big]$$

$$= 1 - m\sum_{\omega\gamma < E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} - m\sum_{\substack{\gamma_1,\gamma_2\\ -(\gamma_1,\gamma_2)_H}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha}$$
$$+A_{\beta\alpha}A^*_{\beta\alpha}(A_{\alpha\alpha} + A^*_{\alpha\alpha} + m(m-2)A_{\alpha\alpha}A^*_{\alpha\alpha}) + \frac{m(7m-29)}{2}A^2_{\alpha\alpha}A^{*2}_{\alpha\alpha} \tag{7.3.28}$$

H entrop□ec Renyi d□nontai apì ton tÔpo 7.3.1.

7.4 To apokl□non mèroc entrop□ac sÔmplexhc

Oi apokl□nontec ìroi kuriarqoÔn sthn entrop□a sÔmplexhc. AkoloÔjwc ja krat soume touc ìrouc autoÔc. Arqik^ or□zoume

$$\Delta \equiv \sum_{\omega_\gamma < E} B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha} + \sum_{\substack{\gamma_1,\gamma_2\\ -(\gamma_1,\gamma_2)_H}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} \tag{7.4.1}$$

Ta pl^th $B_{\beta\gamma,\alpha}B^*_{\beta\gamma,\alpha}$ kai $\Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha}$ apokl□noun sto ìrio pou h enèrgeia twn ekpempìmenwn fwton□wn te□nei sto mhdèn. Ta pl^th $A_{\alpha\alpha}A^*_{\alpha\alpha}$ kai $A_{\beta\alpha}A^*_{\beta\alpha}$ e□nai peperasmèna se ep□pedo dèntrou. H posìthta $A_{\alpha\alpha} + A^*_{\alpha\alpha}$ e□nai ep□shc peperasmènh se t^xh e^{2N} lìgw monadiakìthtac. Tèloc h posìthta $A^2_{\alpha\alpha} + A^{*2}_{\alpha\alpha} = (A_{\alpha\alpha} + A^*_{\alpha\alpha})^2 - 2A_{\alpha\alpha}A^*_{\alpha\alpha}$ e□nai ep□shc peperasmènh.

Me b^sh tic pio p^nw sqèseic, br□skoume ìti

$$\mathrm{Tr}(\rho_H)^m_{singular} = 1 - m\Delta \tag{7.4.2}$$

'Etsi, ìpwc deÐxame sto prohgoÔmeno kef^laio, h apoklÐnousa entropÐa sÔmplexhc dÐnetai apì thn èkfrash

$$S^{m}_{ent,sing} = \Delta = \sum_{\omega\gamma<E} B_{\beta\gamma,\alpha}B^{*}_{\beta\gamma,\alpha} + \sum_{\substack{\gamma_1,\gamma_2 \\ -(\gamma_1,\gamma_2)_H}} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^{*}_{\beta\gamma_1\gamma_2,\alpha} \tag{7.4.3}$$

Qrhsimopoi¸ntac thn sqèsh

$$\sum_{\omega_{\gamma_2}<E}\sum_{\omega_{\gamma_1}<\omega_{\gamma_2}} = \frac{1}{2}\sum_{\omega_{\gamma_2}<E}\sum_{\omega_{\gamma_1}<E} + \frac{1}{2}\sum_{\omega_{\gamma_2}<E}\delta_{\omega_{\gamma_1},\omega_{\gamma_2}} \tag{7.4.4}$$

ìpou o teleutaÐoc ìroc dÐnei amelhtèa suneisfor^, kai paÐrnontac to suneqèc ìrio, brÐskoume

$$\begin{aligned} S^{m}_{ent,sing} = \Delta = &\int_{\lambda}^{E} d\omega_{\gamma}B_{\beta\gamma,\alpha}B^{*}_{\beta\gamma,\alpha} + \int_{\lambda}^{E} d\omega_{\gamma_1}\int_{E} d\omega_{\gamma_2}\Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^{*}_{\beta\gamma_1\gamma_2,\alpha} \\ &+\frac{1}{2}\int_{\lambda}^{E} d\omega_{\gamma_1}\int_{\lambda}^{E} d\omega_{\gamma_2}\Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^{*}_{\beta\gamma_1\gamma_2,\alpha}\Theta(E-\omega_{\gamma_1}-\omega_{\gamma_2}) \end{aligned} \tag{7.4.5}$$

'Eqoume qwrÐsei to ^jroisma wc proc tic katast^seic twn dÔo fwtonÐwn ¸ste na mporoÔme na efarmìsoume ta upèrujra jewr mata.

Epeid o pr¸toc ìroc upologÐzetai se t^xh e^{2N+4}, ja suneisfèroun kai diagr^mmata me ènan eswterikì brìqo. Ta upèrujra jewr mata gia fwtìnia brìqou dÐnoun (kef^laio 3)

$$A_{\beta\alpha}A^{*}_{\beta\alpha} = (A_{\beta\alpha}A^{*}_{\beta\alpha})^{\Lambda}\left(\frac{\lambda}{E}\right)^{2\mathcal{B}_{\beta\alpha}} = (A_{\beta\alpha}A^{*}_{\beta\alpha})^{\Lambda}\left(1 - 2\mathcal{B}_{\beta\alpha}\log\left(\frac{E}{\lambda}\right) + \dots\right) \tag{7.4.6}$$

ìpou h èkfrash $(A_{\beta\alpha}A^{*}_{\beta\alpha})^{\Lambda}$ den perilamb^nei suneisforèc apì upèrujra eikonik^ fwtìnia. UpenjumÐzoume ìti ta upèrujra jewr mata gia ekpempìmena fwtìnia dÐnoun

$$\int_{\lambda}^{E} d\omega_{\gamma}B_{\beta\gamma,\alpha}B^{*}_{\beta\gamma,\alpha} = 2A_{\beta\alpha}A^{*}_{\beta\alpha}\mathcal{B}_{\beta\alpha}\log\left(\frac{E}{\lambda}\right) \tag{7.4.7}$$

Me b^sh ta upèrujra jewr mata, brÐskoume loipìn

$$\begin{aligned} &\int_{\lambda}^{E} d\omega_{\gamma}B_{\beta\gamma,\alpha}B^{*}_{\beta\gamma,\alpha} + \frac{1}{2}\int_{\lambda}^{E} d\omega_{\gamma_1}\int_{\lambda}^{E} d\omega_{\gamma_2}\Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^{*}_{\beta\gamma_1\gamma_2,\alpha}\Theta(E-\omega_{\gamma_1}-\omega_{\gamma_2}) + \mathcal{O}(e^{2N+6}) \\ &= 2(A_{\beta\alpha}A^{*}_{\beta\alpha})^{\Lambda}\left(\mathcal{B}_{\beta\alpha}\log\left(\frac{E}{\lambda}\right) - \mathcal{B}^{2}_{\beta\alpha}\log\left(\frac{E}{\lambda}\right)^{2}\right) \\ &\quad +\mathcal{O}(e^{2N+6}) \end{aligned} \tag{7.4.8}$$

H èkfrash $(A_{\beta\alpha}A^*_{\beta\alpha})^{\Lambda}$ den perilamb^nei upèrujrec apeirÐec kai upologÐzetai se t^xh e^{2N+2}. EpÐshc

$$\int_{\lambda}^{E} d\omega_{\gamma_1} \int_{E} d\omega_{\gamma_2} \Gamma_{\beta\gamma_1\gamma_2,\alpha}\Gamma^*_{\beta\gamma_1\gamma_2,\alpha} = 2\left(\int_{E} d\omega_{\gamma_2} B_{\beta\gamma_2,\alpha}B^*_{\beta\gamma_2,\alpha}\right)\mathcal{B}_{\beta\alpha}\log\left(\frac{E}{\lambda}\right) \tag{7.4.9}$$

Sundu^zontac tic sqèseic 7.4.5-7.4.9, paÐrnoume thn èkfrash gia to apoklÐnon mèroc thc entropÐac sÔmplexhc

$$\begin{aligned} S_{ent,sing} = {} & 2A_{\beta\alpha}A^*_{\beta\alpha}\left(\mathcal{B}_{\beta\alpha}\log\left(\frac{E}{\lambda}\right) - \mathcal{B}^2_{\beta\alpha}\log\left(\frac{E}{\lambda}\right)^2\right) \\ & +2\left(\int_{E} d\omega_{\gamma_2} B_{\beta\gamma_2,\alpha}B^*_{\beta\gamma_2,\alpha}\right)\mathcal{B}_{\beta\alpha}\log\left(\frac{E}{\lambda}\right) \end{aligned} \tag{7.4.10}$$

An kai up^rqoun shm^dia ekjetopoÐhshc parathroÔme ìti ìroi deÔterhc t^xhc suneisfèroun sto suntelest thc pr¸thc dÔnamhc tou logarÐjmou.

EpÐlogoc

Me b^sh ta pio p^nw apotelèsmata sumperaÐnoume ìti sthn telik kat^stash miac skèdashc parousi^zetai sÔmplexh metaxÔ twn uperÔjrwn kai twn uperiwd¸n bajm¸n eleujerÐac. H entropÐa sÔmplexhc eÐnai mh mhdenik kai ekdhl¸nei mia logarijmik apìklish sto ìrio pou o ìgkoc tou sust matoc gÐnetai ^peiroc (toul^qiston sto diataraktikì epÐpedo). Autì apoteleÐ èndeixh ìti oi upèrujroi bajmoÐ eleujerÐac metafèroun plhroforÐa. Epeid h entropÐa sÔmplexhc parousi^zei logarijmikì apeirismì sto suneqèc ìrio, mporeÐ epiprìsjeta upèrujra swmatÐdia na metafèroun mèroc thc plhroforÐac pou leÐpei kat^ thn exaÖlwsh miac maÔrhc trÔpac kai me autìn ton mhqanismì na lÔnetai na amblÔnetai to par^doxo diat rhshc thc plhroforÐac tou Hawking.

Par^rthma A

Tanustèc

Tanustèc eÐnai majhmatik^ antikeÐmena pou orÐzontai se suneqeÐc kampulwmènoc q¸rouc. QarakthrÐzontai apì ton trìpo pou metasqhmatÐzontai k^tw apì genikoÔc metasqhmatismoÔc suntetagmènwn. Oi deÐktec twn tanust¸n qwrÐzontai se 2 kathgorÐec: touc p^nw deÐktec pou onom^zontai antalloÐwtoi kai touc k^tw pou onom^zontai sunalloÐwtoi.

'Estw ìti se èna sÔsthma to opoÐo perigr^fetai apì tic suntetagmènec x^α orÐzetai ènac tanust me n p^nw deÐktec kai m k^tw deÐktec

$$T^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_m} \tag{A.1}$$

Tìte o tanust c se èna ^llo sÔsthma me suntetagmènec x'^α ja dÐnetai apì thn èkfrash

$$T'^{\alpha_1\alpha_2\ldots\alpha_n}_{\beta_1\beta_2\ldots\beta_n} = \frac{\partial x'^{\alpha_1}}{\partial x^{\sigma_1}}\ldots\frac{\partial x'^{\alpha_n}}{\partial x^{\sigma_n}}\frac{\partial x^{\rho_1}}{\partial x'^{\beta_1}}\ldots\frac{\partial x^{\rho_m}}{\partial x'^{\beta_m}}T^{\sigma_1\sigma_2\ldots\sigma_n}_{\rho_1\rho_2\ldots\rho_n} \tag{A.2}$$

H eidik perÐptwsh tanust me ènan p^nw deÐkth onom^zetai antalloÐwto di^nusma kai me ènan k^tw deÐkth sunalloÐwto di^nusma. Tanustèc me kajìlou deÐktec eÐnai bajmwtèc sunart seic pou paramènoun analloÐwtec k^tw apì genikoÔc metasqhmatismoÔc suntetagmènwn. H par^gwgoc miac bajmwt c posìthtac par^gei èna sunalloÐwto di^nusma: $\partial_\alpha\Phi$. Ektìc apì tic bajmwtèc sunart seic amet^blhta k^tw apì genikoÔc metasqhmatismoÔc suntetagmènwn paramènoun o pl rwc antisummetrikìc tanust c Levi-Civita kai o mhdenikìc tanust c tou ìpoiou ìla ta stoiqeÐa isoÔntai me mhdèn.

MetaxÔ tanust¸n orÐzontai treic pr^xeic:

Upèrjesh

$$T^{\alpha\beta}_{\ \ \gamma} = aS^{\alpha\beta}_{\ \ \gamma} + bP^{\alpha\beta}_{\ \ \gamma} \tag{A.3}$$

ìpou a,b eÐnai arijmoÐ

Exwterikì ginìmeno

$$T^{\alpha\beta}_{\ \ \gamma} = aS^{\alpha}P^{\beta}_{\ \gamma} \tag{A.4}$$

Sustol

$$T^{\alpha} \equiv T^{\alpha\beta}_{\ \ \beta} \tag{A.5}$$

Oi pio p^nw idiìthtec isqÔoun gia aujaÐreto arijmì deikt¸n kai mporoÔn na sunduastoÔn me aujaÐreto trìpo.

E^n ènac tanust c isoÔtai me ton mhdenikì tanust se èna sÔsthma, tìte ja isoÔtai me ton mhdenikì tanust se ìla ta sust mata.

H par^gwgoc enìc tanust den eÐnai tanust c. 'Etsi orÐzoume thn sunalloÐwth par^gwgo ¸ste h dr^sh thc se èna tanust na dÐnei ènan tanust .

SunalloÐwth par^gwgoc

$$T^{\alpha}\ ;\nu \equiv \frac{\partial T^{\alpha}}{\partial x^{\nu}} + \Gamma^{\alpha}_{\ \nu\lambda}T^{\lambda}, \quad T_{\alpha;\nu} \equiv \frac{\partial T_{\alpha}}{\partial x^{\nu}} - \Gamma^{\lambda}_{\ \nu\alpha}T_{\lambda} \tag{A.6}$$

ìpou $\Gamma^{\alpha}_{\ \mu\nu}$ h sÔndesh. H sunalloÐwth par^gwgoc tanust¸n me perissìterouc deÐktec paÐrnei èna ìro an^logo thc sÔndeshc, ìpwc pio p^nw, gia k^je deÐkth. H sunalloÐwth par^gwgoc eÐnai grammikìc telest c kai upakoÔei ston nìmo tou Leibniz.

Parˆrthma B

Genik sqetikìthta

H jewrÐa thc genik c sqetikìthtac perigrˆfei thn barÔthtac sto klassikì epÐpedo. Se autì to kefˆlaio ja qrhsimopoi soume tic arqèc thc genik c sqetikìthtac gia na prosdiorÐsoume tic exis¸seic tou Eistein kai tic exis¸seic pou prosdiorÐzoun thn troqiˆ enìc s¸matoc pou kineÐtai eleÔjera se barutikì pedÐo.

B.1 Exis¸seic Eistein

Me bˆsh thn Arq thc Genik c Sqetikìthtac *μια εξίσωση ισχύει στην παρουσία βαρύτητας, εάν η εξίσωση είναι συμβατή με τις αρχές της ειδικής σχετικότητας στην απουσία βαρύτητας και δεν αλλάζει κάτω από γενικούς μετασχηματισμούς συντεταγμένων.*

Mia exÐswsh pou paramènei ametˆblhth kˆtw apì genikoÔc metasqhmatismoÔc suntetagmènwn isqÔei se ìla ta sust mata anaforˆc suntetagmènwn. SÔmfwna me thn arq thc isodunamÐac tou Eistein mporoÔme topikˆ na broÔme èna adraneiakì sÔsthma anaforˆc, sto opoÐo oi epidrˆseic thc barÔthtac anairoÔntai (topikˆ) kai isqÔoun oi nìmoi thc eidik c sqetikìthtac. Epomènwc h arq c thc genik c sqetikìthtac aporrèei apì thn arq thc isodunamÐac tou Eistein. AkoloÔjwc parousiˆzontai se suntomÐa oi exis¸seic tou Eistein [5, 28].

OrÐzoume wc apeirostì diˆsthma thn analloÐwth posìthta

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{B.1.1}$$

ìpou $g_{\mu\nu}$ h metrik pou kajorÐzei th gewmetrÐa tou q¸rou. H sÔndesh dÐnetai sunart sei thc metrik c apì thn akìloujh sqèsh

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\lambda} \left(\frac{\partial g_{\nu\lambda}}{\partial x^\mu} + \frac{\partial g_{\mu\lambda}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right) \tag{B.1.2}$$

O tanust c kampulìthtac Riemann $R^{\alpha}{}_{\beta\gamma\delta}$ ikanopoieÐ thn akìlouj h sqèsh gia opoiod pote tetradiˆnusma V_α:

$$R^{\alpha}{}_{\beta\gamma\delta}V_\alpha = V_{\alpha;\beta;\gamma} - V_{\alpha;\gamma;\beta} \tag{B.1.3}$$

Eˆn ìlec oi sunist¸sec tou tanust Riemann eÐnai mhdèn, tìte o q¸roc eÐnai epÐpedoc. Eˆn ìqi tìte o q¸roc eÐnai kampulwmènoc. O tanust c Riemann loipìn qarakthrÐzei thn kampÔlwsh tou q¸rou.

O mìnoc tanust c pou prokÔptei apì sustol dÔo deikt¸n tou tanust Riemann eÐnai o tanust c Ricci

$$R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta} \tag{B.1.4}$$

PaÐrnontac to Ðqnoc tou tanust Ricci prokÔptei h bajmwt kampulìthta Ricci

$$R = R^{\alpha}{}_{\alpha} \tag{B.1.5}$$

Epeid o tanust c Riemann perigrˆfei pl rwc thn kampulìthta tou qwrìqronou kai exartˆtai mìno apì thn metrik , tic pr¸tec kai tic deÔterec parag¸gouc thc, tìte h exÐswsh pou yˆqnoume ja perièqei thn metrik , ton tanust Ricci, thn bajmwt kampulìthta Ricci kai ton tanust enèrgeiac-orm c afoÔ phg thc barÔthtac eÐnai h enèrgeia. Lambˆnontac upìyin autˆ kai thn arq thc genik c sqetikìthta katal goume sthn exÐswsh tou barutikoÔ pedÐou tou Einstein

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{B.1.6}$$

ìpou L h kosmologik stajerˆ. H lÔsh twn exis¸sewn aut¸n dÐnei th metrik pou prosdiorÐzei thn gewmetrÐa tou qwrìqronou. H exÐswsh aut prokÔptei apì thn drˆsh Eistein-Hilbert

$$S_{EH} = \int d^D x\sqrt{|g|}(R - 2\Lambda) + S_{matter} \tag{B.1.7}$$

ìpou g h orÐzousa thc metrik c. H S_{matter} eÐnai h drˆsh thc Ôlhc kai thc aktinobolÐac, me bˆsh thn opoÐa prosdiorÐzoume ton tanust enèrgeiac-orm c

$$T^{\mu\nu} = -2\frac{\delta S_{matter}}{\delta g_{\mu\nu}} \tag{B.1.8}$$

Basizìmenoi sthn arq thc genik c sqetikìthtac mporoÔme na genikeÔsoume kˆje exÐswsh pou isqÔei sthn eidik sqetikìthta ¸ste na isqÔei sthn parousÐa barÔthtac. Prèpei na antikatast soume tic merikèc parag¸gouc me sunalloÐwtec parag¸gouc kai th metrik Minkowski me th genik metrik sthn parousÐa barÔthtac. O stoiqei¸dhc ìgkoc $d^D x$

antikajÐstatai me ton ìgko $d^D x\sqrt{|g|}$, o opoÐoc paramènei ametˆblhtoc kˆtw apì genikoÔc metasqhmatismoÔc suntetagmènwn.

B.2 Exis¸seic kÐnhshc swmatidÐwn

H eleÔjerh kÐnhsh twn swmatidÐwn se èna barutikì pedÐo perigrˆfetai apì thn lagkrazian [28]

$$L = \sqrt{-g_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\mu}}{d\sigma}} \tag{B.2.1}$$

h opoÐa odhgeÐ sthn exÐswsh twn gewdaitik¸n kampul¸n

$$\frac{d^2x^{\alpha}}{d\sigma^2} + \Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{d\sigma}\frac{dx^{\nu}}{d\sigma} = 0 \tag{B.2.2}$$

Ja mporoÔsame wc parˆmetro na qrhsimopoi soume ton idiìqrono τ, gia ton opoÐo isqÔei $d\tau^2 = -dS^2$. Tìte ja mporoÔsame na qrhsimopoi soume thn L^2 antÐ thc L, kaj¸c odhgeÐ stic Ðdiec exis¸seic kÐnhshc. Oi gewdaitikèc kampÔlec antiproswpeÔoun tic 'eujeÐec' grammèc stouc kampulwmènouc qwrìqronouc. H barÔthta kai h kampÔlwshc tou qwrìqronou eÐnai to Ðdio prˆgma.

Par ˆ r thma C

Kbantik jewrÐa pedÐou

PedÐa mporoÔn na jewrhjoÔn ì lec oi qwroqronoexarthmènec sunart seic stic opoÐec mporoÔme na apod¸soume k ˆpoia fusik shmasÐa. Upˆrqoun tèssera eÐdh pedÐwn, ta bajmwtˆ pedÐa (ìtan to pedÐo eÐnai mia bajmwt sunˆrthsh), ta dianusmatikˆ pedÐa, ta spinoriakˆ pedÐa kai ta tanustikˆ pedÐa. H diˆkrish aut twn pedÐwn ofeÐletai sto p¸c metasqhmatÐzontai kˆtw apì metasqhmatismoÔc Lorentz. Gia parˆdeigma ta bajmwtˆ pedÐa paramènoun ametˆblhta kˆtw apì metasqhmatismoÔc Lorentz, kai ta dianusmatikˆ pedÐa metasqhmatÐzontai san tetradianÔsmata. AkolouÔjwc ja exetˆsoume èna parˆdeigma bajmwtoÔ pedÐou (Klein-Gordon) kai èna parˆdeigma spinoriakoÔ pedÐou me 4 migadikèc sunist¸sec (pedÐo Dirac). Stìqoc mac eÐnai na doÔme pwc exˆgontai oi kanìnec Feynman gia thn kbantik hlektrodunamik (QED).

C.1 Bajmwtˆ pedÐa

'Estw èqoume èna bajmwtì pedÐo $\varphi(x)$ ìpou to x eÐnai to antalloÐwto tetradiˆnisma tou q¸rou kai tou qrìnou. H lagkrazian puknìthta tou sust matoc sundèetai me thn lagkrazian mèsw thc sqèshc

$$L(\varphi, \partial_\mu \varphi) = \int d^3\vec{x}\, \mathcal{L}\, (\varphi, \partial_\mu \varphi) \tag{C.1.1}$$

ìpou L h lagkrazian kai $\mathcal{L}$ h lagkrazian puknìthta (thn ìpoia ja anafèroume wc aplˆ lagkrazian gia skopoÔc aplìthtac). H drˆsh tou sust matoc dÐnetai apì thn sqèsh

$$S(\varphi, \partial_\mu \varphi) = \int d^4x\, \mathcal{L}\, (\varphi, \partial_\mu \varphi) \tag{C.1.2}$$

Elaqistopoi¸ntac thn drˆsh krat¸ntac ta pedÐa sta sÔnora stajerˆ paÐrnoume tic exis¸seic Euler-Lagrange

$$\frac{\partial L}{\partial \varphi} - \partial_\mu \left[\frac{\partial L}{\partial(\partial_\mu \varphi)} \right] = 0 \tag{C.1.3}$$

H puknìthta orm c orÐzetai apì thn èkfrash

$$\pi(x) = \frac{\partial L}{\partial \dot{\varphi}} \quad \dot{\varphi} = \frac{\partial \varphi}{\partial t} \tag{C.1.4}$$

kai h puknìthta enèrgeiac orÐzetai wc

$$H = \pi(x)\dot{\varphi} - L \tag{C.1.5}$$

Sthn perÐptwsh pou upˆrqoun pollˆ pedÐa, gia kˆje pedÐo antistoiqeÐ mia puknìthta orm c kai sthn èkfrash gia thn puknìthta enèrgeiac o pr¸toc ìroc antikajÐstatai me ˆjroisma tètoiwn ìrwn, èna gia kˆje pedÐo. EpÐshc sthn perÐptwsh poll¸n pedÐwn upˆrqei mia exÐswsh Euler-Lagrange gia kˆje pedÐo.

C.2 Kanonik kbˆntwsh

H kanonik kbˆntwsh perilambˆnei 2 b mata:

1: Proˆgoume ìlec tic klasikèc posìthtec se telestèc.

$$\varphi(\vec{x},t) \to \widehat{\varphi}(\vec{x},t) \ , \ \pi(\vec{x},t) \to \widehat{\pi}(\vec{x},t) \tag{C.2.1}$$

2: ApaitoÔme oi telestèc na upakoÔn stic kanonikèc sqèseic metˆjeshc se Ðsouc qrìnouc:

$$[\widehat{\varphi}(\vec{x},t), \widehat{\pi}(\vec{y},t)] = i\delta^d(\vec{x}-\vec{y}) \tag{C.2.2}$$

$$[\widehat{\varphi}(\vec{x},t), \widehat{\varphi}(\vec{y},t)] = 0, \ [\widehat{\pi}(\vec{x},t), \widehat{\pi}(\vec{y},t)] = 0 \tag{C.2.3}$$

ìpou d eÐnai o arijmìc twn qwrik¸n diastˆsewn tou q¸rou ston opoÐo brÐsketai to pedÐo[1]. 'Epeita ekfrˆzoume ta pedÐa kai ìlec tic upìloipec fusikèc posìthtec (ìpwc h qamiltonian) wc sunart seic telest¸n dhmiourgÐac kai katastrof c. Oi telestèc autoÐ dhmiourgoÔn kai katastrèfoun swmatÐdia ìtan drˆsoun se kˆpoia katˆstash. Ta swmatÐdia eÐnai kbˆnta me sugkekrimènh enèrgeia kai orm .

[1] Στην περίπτωση του πεδίου Dirac επιβάλλουμε σχέσεις αντιμετάθεσης αντί μετάθεσης.

C.3 Kbˆntwsh mèsw sunarthsiak¸n oloklhrwmˆtwn

O deÔteroc kai pio qr simoc trìpoc kbˆntwshc mia klasik c jewrÐac eÐnai mèsw sunarthsiak¸n oloklhrwmˆtwn. Aut h mèjodoc basÐzetai sta sunarthsiakˆ oloklhr¸mata Feynman pou eÐnai ènac enallaktikìc trìpoc perigraf c thc kbantik c mhqanik c. EÐnai h pio shmantik apì tic dÔo mejìdouc kai eÐnai aparaÐthth se peript¸seic pou h mèjodoc thc kanonik c kbˆntwshc eÐnai polÔ dÔskolo adÔnato na efarmosteÐ ìpwc sthn QCD kai sthn jewrÐa qord¸n [2]. To basikì pleonèkthma thc eÐnai ìti gia na d¸sei apotelèsmata den qreiˆzetai na diagwniopoihjeÐ h qamiltonian (dhlad na brejoÔn oi idiokatastˆseic thc).

H pijanìthta metˆbashc enìc kbantikoÔ sust matoc apì mia arqik katˆstash $|q_i,t_i\rangle$ se mia telik katˆstash $|q_f,t_f\rangle$ ekfrˆzetai apì ton diadìth

$$K(q_f,t_f;fq_i,t_i) = \langle q_f,t_f|q_i,t_i\rangle = \int [dq]_{q_i,t_i}^{q_f,t_f}\; e^{\frac{iS[q]}{\hbar}} \tag{C.3.1}$$

ìpou h olokl rwsh gÐnetai wc proc ìla ta pijanˆ monopˆtia me arq q_i kai tèloc q_f. Kˆje tètoio monopˆti suneisfèrei anˆloga me ton ekjetikì parˆgonta $e^{\frac{iS[q]}{\hbar}}$ ìpou $S[q]$ eÐnai h drˆsh tou monopatioÔ. Eˆn to q antiproswpeÔei mia suntetagmènh jèshc tìte sto klasikì ìrio ìpou $\hbar \to 0$, exaitÐac tou ekjetikoÔ parˆgonta, geitonikˆ monopˆtia ja suneisfèroun me polÔ diaforetikì trìpo, me apotèlesma na allhloanaireÐtai h suneisforˆ touc. Oi mìnec suneisforèc pou ja epibi¸noun eÐnai gia monopˆtia kontˆ sto elˆqisto thc drˆshc, afoÔ tètoia monopˆtia èqoun Ðdio ekjetikì parˆgontac (lìgw tou ìti h parˆgwgoc thc drˆshc mhdenÐzetai sto elˆqisto thc). 'Etsi paÐrnoume to klasikì apotèlesma, dhlad to klasikì sÔsthma akoloujeÐ to monopˆti pou elaqistopoieÐ thn drˆsh tou.

H anamenìmenh tim tou telest $q(t)$ dÐnetai apì thn èkfrash

$$\langle q(t)\rangle = \langle q_f,t_f|q(t)|q_i,t_i\rangle = \int [dq]_{q_i,t_i}^{q_f,t_f}\; q(t)e^{\frac{iS[q]}{\hbar}}, \quad t_i<t<t_f \tag{C.3.2}$$

Eˆn jèloume thn anamenìmenh tim tou ginomènou dÔo telest¸n $q(t),q(t')$, lìgw tou ìti autoÐ oi telestèc sthn eikìna Heisenberg emplèkoun thn qamhltonian , den metatÐjentai kai ètsi prèpei na lˆboume upìyin thn qronik seirˆ touc. H posìthta

$$\left\langle q(t)q(t')\right\rangle = \langle q_f,t_f|q(t)q(t')|q_i,t_i\rangle \tag{C.3.3}$$

eÐnai to plˆtoc metˆbashc apì thn arqik katˆstash sthn telik katˆstash me thn parembol 2 drˆsewn tou telest q, epomènwc h perÐptwsh ìpou $t<t'$ den suneisfèrei afoÔ o

[2]QCD = κβαντική χρωμοδυναμική.

qrìnoc den mporeÐ na p^ei antÐstrofa. 'Etsi orÐzoume ton telest qronik c seir^c T

$$\begin{aligned} T[q(t)q(t^{'})] =& q(t)q(t^{'}) \ \gamma\iota\alpha \ t > t^{'} \\ =& q(t^{'})q(t) \ \gamma\iota\alpha \ t < t^{'} \end{aligned} \tag{C.3.4}$$

'Omoia gia aujaÐreto arijmì telest¸n.

C.3.1 EukleÐdeia sunarthsiak^ oloklhr¸mata

Metasqhmatèzoume ton qrìno t ston eukleÐdeio qrìno u mèsw thc sqèshc

$$t = -iu \tag{C.3.5}$$

kai ètsi o 4-di^statoc qwrìqronoc Minkowski metasqhmatÐzetai ston 4-di^stato eukleÐdeio q¸ro. Jètontac $t_i = 0$ ta sunarthsiak^ oloklhr¸mata (sunart seic susqètishc) paÐrnoun thn mor f

$$\langle q_f, U| \, T(\prod_a \widehat{q}(u_a)) \, |q_i, 0\rangle_E = \int [dq]_{q_i,0}^{q_f,U} \, e^{-\frac{S_E}{\hbar}} \tag{C.3.6}$$

ìpou S_E eÐnai h eukleÐdeia dr^sh pou dÐnetai apì thn èkfrash:

$$S_E = \int du \left[\frac{1}{2} m \left(\frac{dq}{dt_E} \right)^2 + V(q) \right] \tag{C.3.7}$$

C.3.2 Par^deigma armonikoÔ talantwt

Wc efarmog twn sunarthsiak¸n oloklhrwm^twn ja upologÐsoume pl^th met^bashc metaxÔ katast^sewn tou armonikoÔ talantwt qwrÐc na diagwniopoi soume thn qamiltonian [29].

To eukleÐdeio pl^toc met^bashc dÐnetai apì thn èkfrash

$$\langle q_f, U|q_i, 0\rangle = \int [dq]_{q_i,0}^{q_f,U} \, e^{-S_E} \tag{C.3.8}$$

H eukleÐdeia dr^sh isoÔtai me

$$S_E = \frac{1}{2} m \int_0^U du \big[(\partial_u q)^2 + \omega^2 q^2 \big] + S_{ct} \tag{C.3.9}$$

H S_{ct} omalopoieÐ thn jewrÐa, dhlad aporrof^ touc apeirismoÔc pou prokÔptoun apì thn S_E, kai ja prosdioristeÐ argìtera. All^zoume thn metablht tou sunarthsiakoÔ oloklhr¸matoc

$$q(u) = q_{cl}(u) + q'(u) \tag{C.3.10}$$

ìpou q_{cl} eÐnai to klassikì monop^ti kai upakoÔei stic exis¸seic tou NeÔtwna

$$-\frac{d^2 q_{cl}(u)}{du^2} + \omega^2 q_{cl}(u) = 0, \quad q_{cl}(0) = q_i, \quad q_{cl}(U) = q_f \tag{C.3.11}$$

kai

$$q'(0) = 0, \quad q'(U) = 0 \tag{C.3.12}$$

AfoÔ h metablht q sqetÐzetai me thn metablht q' mèsw miac stajer c metatìpishc h sqetik iakwbian metaxÔ twn dÔo metablht¸n eÐnai mon^da kai mporoÔme na antikatast - soume to mètro $\int [dq]^{q_f,U}_{q_i,0}$ me to mètro $\int [dq']^{0,U}_{0,0}$. 'Etsi h dr^sh gr^fetai sthn mor f

$$S_E = S_{cl} + S' + m \int du \big[\partial_u q' \partial_u q_{cl} + \omega^2 q_{cl} q'\big] + S_{ct} \tag{C.3.13}$$

ìpou

$$S' = \frac{1}{2} m \int_0^U du \big[(\partial_u q')^2 + \omega^2 q'^2\big] \tag{C.3.14}$$

kai

$$S_{cl} = \frac{1}{2} m \int_0^U du \big[(\partial_u q_{cl})^2 + \omega^2 q_{cl}^2\big] = \frac{\omega}{2\sinh(\omega U)} \big[(q_i^2 + q_f^2)\cosh(\omega U) - 2 q_i q_f\big] \tag{C.3.15}$$

O proteleutaÐoc ìroc mhdenÐzetai me b^sh tic exis¸seic kÐnhshc. 'Etsi to pl^toc met^bashc paÐrnei thn mor f

$$\langle q_f, U | q_i, 0\rangle = e^{-S_E(q_i,q_f) - S_{ct}} \int [dq']^{0,U}_{0,0} e^{-S'} \tag{C.3.16}$$

Gia na upologÐsoume sunarthsiak^ oloklhr¸mata thc mor f c

$$\int [d\Phi] e^{-S_E} \tag{C.3.17}$$

gr^foume ton ekjèthc sthn mor f

$$S_E = \frac{1}{2} \int d^n x \Phi(x) \Delta \Phi(x) \tag{C.3.18}$$

ìpou Δ ènac diaforikìc telest c. 'Estw ìti o telest c Δ èqei idiokatast^seic φ_i me idiotimèc λ_i pou apoteloÔn pl rh orjokanonik b^sh ston q¸ro twn sunart sewn me tic sugkekrimènec sunoriakèc sunj kec

$$\Delta\Phi_i = \lambda_i \Phi_i, \quad \Phi = \sum_i \phi_i \Phi_i, \quad \int d^n x \Phi_i(x)\Phi_j(x) = \delta_{ij} \tag{C.3.19}$$

ìpou ϕ_i migadikoÐ suntelestèc. Tìte to sunarthsiakì olokl rwma mporeÐ na grafteÐ sthn mor f

$$\int [d\Phi] e^{-\frac{1}{2}\int d^n x \Phi(x)\Delta\Phi(x)} = \int \left(\prod_i d\Phi_i\right) e^{-\sum_{i,j}\int \phi_i \Phi_i(x)\Delta\phi_j\Phi_j(x)} = \int \left(\prod_i d\Phi_i e^{-\frac{1}{2}\lambda_i \Phi_i^2}\right) \tag{C.3.20}$$

H teleutaÐa èkfrash eÐnai ginìmeno gkaousian¸n oloklhrwm˘twn kai dÐnetai apì thn èkfrash

$$\int \left(\prod_i d\Phi_i e^{-\frac{1}{2}\lambda_i \Phi_i^2}\right) = \prod_i \left(\frac{2\pi}{\lambda_i}\right)^{\frac{1}{2}} = \det\left(\frac{\Delta}{2\pi}\right)^{-\frac{1}{2}} \tag{C.3.21}$$

ìpou o telest c D eÐnai ekfrasmènoc se mor f pÐnaka me stoiqeÐa

$$\Delta_{ij} = \langle \Phi_i | \hat{\Delta} | \Phi_j \rangle \tag{C.3.22}$$

Sthn perÐptwsh tou armonikoÔ talantwt èqoume (sÔmfwna me tic ekfr˘seic C.3.14 kai C.3.16)

$$S_E \to S' = \frac{1}{2} m \int_0^U du \big[(\partial_u q')^2 + \omega^2 q'^2\big] = \frac{1}{2} m \int_0^U du q' \Big(-\frac{\partial^2}{\partial u^2} + \omega^2\Big) q' \tag{C.3.23}$$

Epomènwc

$$\Delta = -\frac{\partial^2}{\partial u^2} + \omega^2 \tag{C.3.24}$$

Lamb˘nontac upìyin tic periodikèc sunj kec, $q'(0) = q'(U) = 0$, brÐskoume tic idiosunart sic tou diaforikoÔ telest kai tic antÐstoiqec idiotimèc

$$f_i(u) = \left(\frac{2}{U}\right)^{\frac{1}{2}} \sin\left(\frac{i\pi u}{U}\right), \quad \lambda_i = \frac{i^2\pi^2}{U^2} + \omega^2 \tag{C.3.25}$$

kai ètsi

$$\det\left(\frac{\Delta}{2\pi}\right) = \prod_i^{\infty} \frac{i^2\pi^2 + \omega^2 U^2}{2\pi U^2} \tag{C.3.26}$$

To ^peiro ginìmeno sthn teleutaÐa èkfrash apoklÐnei. Gia na to omalopoi soume to diairoÔme me èna talantwt meg^lhc suqnìthtac W

$$\det\left(\frac{\Delta}{2\pi}\right) \to \prod_{i}^{\infty} \frac{i^2\pi^2 + \omega^2 U^2}{i^2\pi^2 + \Omega^2 U^2} = \frac{\Omega \sinh(\omega U)}{\omega \sinh(\Omega U)} \tag{C.3.27}$$

Sto teleutaÐo b ma qrhsimopoi same thn èkfrash

$$\prod_{k=1}^{\infty} \frac{k+iz}{k}\frac{k-iz}{k} = \frac{1}{\pi z}\sinh(\pi z) \tag{C.3.28}$$

Gia mikr^ i h èkfrash diafèrei apì thn arqik èkfrash wc proc èna pollaplasiastikì par^gonta, en¸ gia meg^la i to ginìmeno sugklÐnei. To pl^toc met^bashc gÐnetai

$$\langle q_f, U | q_i, 0\rangle = e^{-S_E(q_i,q_f) - S_{ct}} \int [dq']_{0,0}^{0,U} e^{-S'} = \lim_{\Omega\to\infty} e^{-S_E(q_i,q_f)-S_{ct}} \left(\frac{\omega \sinh(\Omega U)}{\Omega \sinh(\omega U)}\right)^{\frac{1}{2}} \tag{C.3.29}$$

Krat¸ntac ìrouc pou den mhdenÐzontai sto ìrio $\Omega \to \infty$, paÐrnoume

$$\langle q_f, U | q_i, 0\rangle = e^{-S_{cl}(q_i,q_f) + \frac{1}{2}(\Omega U - \log\Omega) - S_{ct}} \left(\frac{\omega}{2\sinh(\omega U)}\right)^{\frac{1}{2}} \tag{C.3.30}$$

Gia na broÔme thn morf thc S_{ct} prèpei na sugkrÐnoume to apotèlesma C.3.30 me mia gnwst èkfrash. Gia ton skopì autì paÐrnoume to ìrio $U \to \infty$:

$$\lim_{U\to\infty} \langle q_f, U | q_i, 0\rangle = \langle q_f | 0\rangle \langle 0 | q_i\rangle\, e^{-E_0 U} = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\omega}{2}(q_f^2 + q_i^2 + U)} \tag{C.3.31}$$

ìpou E_0 eÐnai h enèrgeia thc jemeli¸douc kat^stashc kai $|0\rangle$ eÐnai h basik kat^stash tou armonikoÔ talantwt . To pio p^nw apotèlesma prokÔptei e^n paremb^loume mia pl rh b^sh idiokatast^sewn thc enèrgeiac. Sto ìrio $U \to \infty$ epibi¸nei mìno h kat^stash el^qisthc enèrgeiac.

Sto ìrio autì h èkfrash C.3.30 an^getai sthn akìlouqh

$$\lim_{U\to\infty} e^{-S_{cl}(q_i,q_f) + \frac{1}{2}(\Omega U - \log\Omega)\S_{ct}} \left(\frac{\omega}{2\sinh(\omega U)}\right)^{\frac{1}{2}} = \omega^{\frac{1}{2}} e^{-\frac{\omega}{2}(q_f^2+q_i^2+U) + \frac{1}{2}(\Omega U - \log\Omega) - S_{ct}} \tag{C.3.32}$$

SugkrÐnontac brÐskoume

$$S_{ct} = \frac{1}{2}\int_0^U du\Omega - \frac{1}{2}\log\left(\frac{\Omega}{\pi}\right) \tag{C.3.33}$$

To pl˘toc met˘bashc dÐnetai apì thn èkfrash

$$\langle q_f, U|f_i, 0\rangle = \left(\frac{\omega}{2\pi\sinh(\omega U)}\right)^{\frac{1}{2}} e^{-S_{cl}} \tag{C.3.34}$$

C.4 Je¸rhma thc Noether

To je¸rhma thc Noether sundèei diathroÔmenec posìthtec me suneqeÐc summetrÐec.

Ac jewr soume ton akìloujo metasqhmatismì

$$\varphi(x) \to \varphi^{'}(x) = \varphi(x) + \alpha\Delta\varphi(x) + \mathcal{O}(\alpha^2) \tag{C.4.1}$$

ìpou α apeirost˘ mikr par˘metroc, ¸ste na mporoÔme na ameloÔme ìrouc deÔterhc t˘xhc wc proc α. H lagkrazian metasqhmatÐzetai sthn akìloujh

$$\mathcal{L} \to \mathcal{L}^{'} = \mathcal{L} + \alpha\Delta\mathcal{L} + \mathcal{O}(\alpha^2) \tag{C.4.2}$$

H par˘metroc α paÐrnei suneqeÐc timèc.

E˘n o pio p˘nw metasqhmatismìc apoteleÐ summetrÐa tou sust matoc, ìte prèpei na af nei tic exis¸seic kÐnhshc analloÐwtec. IsodÔnama h dr˘sh prèpei na paramènei analloÐwth (na all˘zei kat˘ mia stajer tim , anex˘rthth tou pedÐou). Sunep¸c

$$\Delta S = 0 \Rightarrow \Delta\mathcal{L} = \partial_\mu \mathcal{J}^\mu \tag{C.4.3}$$

ìpou to $\mathcal{J}^\mu$ mia dianusmatik sun˘rthsh. An to pedÐo $\varphi(x)$ ikanopoieÐ tic exis¸seic kÐnhshc (exÐswsh C.1.3) brÐskoume

$$\Delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi\right) \tag{C.4.4}$$

Afair¸ntac tic exis¸seic C.4.3 kai C.4.4, brÐskoume ìti to diathroÔmeno reÔma eÐnai

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta\varphi - \mathcal{J}^\mu \tag{C.4.5}$$

kai ikanopoieÐ thn exÐswsh sunèqeiac

$$\partial_\mu j^\mu = 0 \tag{C.4.6}$$

Gia k˘je suneq summetrÐa thc lagkrazian c prokÔptei èna tètoio reÔma (gnwstì wc reÔma Noether). Apì thn exÐswsh C.4.6 mporoÔme na broÔme diathroÔmeno fortÐo kaj¸c h posìthta $\int d^3\vec{x} j^0$ den metab˘lletai qronik˘ (ìpwc aporrèei apì to je¸rhma Gauss).

C.4.1 Tanust c enèrgeiac-orm c

Mia shmantik efarmog tou jewr matoc eÐnai gia qwroqronikèc metaforèc $x'^{\mu} \to x^{\mu} + \alpha^{\mu}$. Up^rqoun tèsseric anex^rthtec metaforèc, epomènwc anamènoume na broÔme tèssera anex^rthta metaxÔ touc diathroÔmena reÔmata. Wc proc èna metasqhmatismì metafor^c to pedÐo metab^lletai wc akoloÔjwc: $\varphi(x) \to \varphi(x+\alpha) = \varphi(x) + \alpha^{\mu}\partial_{\mu}\varphi(x)$. Me b^sh to je¸rhma thc Noether (efìson oi metasqhmatismoÐ autoÐ apoteloÔn summetrÐa tou sust matoc) prokÔptoun ta tèssera diathroÔmena reÔmata (èna gia k^je qwroqronik sunist¸sa) kai sunapoteloÔn èna tanust me dÔo deÐktec:

$$T^{\mu}{}_{\nu} = \frac{\partial L}{\partial(\partial_{\mu}\varphi)}\partial_{\nu}\varphi - \delta^{\mu}{}_{\nu} L \tag{C.4.7}$$

BrÐskontac thn qamiltonian puknìthta kai qrhsimopoi¸ntac thn exÐswsh C.1.5 sumperaÐnoume ìti sumpÐptei me thn sunist¸sa $T^{0}{}_{0}$ tou pio p^nw tanust . 'Omoia oi sunist¸sec $T^{0}{}_{i}$ sumpÐptoun me tic puknìthtec orm c tou sust matoc. Epomènwc katal goume sto sumpèrasma ìti o pio p^nw tanust c eÐnai o tanust c enèrgeiac-orm c tou sust matoc.

C.5 PedÐo Klein-Gordon

C.5.1 Klasikì pedÐo Klein-Gordon

'Ena bajmwtì pedÐo me thn ex c lagkrazian

$$L = \frac{1}{2}(\partial_{\mu}\varphi)^2 - \frac{1}{2}m^2\varphi^2 \tag{C.5.1}$$

ikanopoieÐ thn akìloujh exÐswsh kÐnhshc

$$(\partial^{\mu}\partial_{\mu} + m^2)\varphi = 0 \tag{C.5.2}$$

H exÐswsh C.5.2 onom^zetai exÐswsh Klein-Gordon. H puknìthta orm c kai enèrgeiac tou pedÐou eÐnai antÐstoiqa

$$\pi(x) = \dot{\varphi} \tag{C.5.3}$$

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2}m^2\varphi^2 \tag{C.5.4}$$

ìpou ∇ o telest c klÐshc. H lÔsh thc exÐswshc C.5.2 dÐnetai apì thn epallhlÐa

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\left[A_{\vec{p}}\, e^{-i\omega_{\vec{p}}t+i\vec{p}\cdot\vec{x}} + B_{\vec{p}}\, e^{i\omega_{\vec{p}}t+i\vec{p}\cdot\vec{x}}\right] \tag{C.5.5}$$

C.5.2 Kb˘ntwsh pedÐou Klein-Gordon

Gia thn kb˘ntwsh tou pedÐou Klein-Gordon prowjoÔme se telestèc to pedÐo $\varphi(x) \to \widehat{\varphi}(x)$ (pediakìc telest c) kai thn puknìthta orm c $\pi(x) \to \widehat{\pi}(x)$ (telest c puknìthtac orm c), ètsi ¸ste na ikanopoioÔn thn sqèsh met˘ jeshc

$$[\widehat{\varphi}(\vec{x},t), \widehat{\pi}(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \tag{C.5.6}$$

Eis˘goume epÐshc touc telestèc $\widehat{\alpha}_{\vec{p}}$ (telest c katastrof c), $\widehat{\alpha}^{\dagger}_{\vec{p}}$ (telest c dhmiourgÐac), kai epib˘lloume tic pio k˘tw sqèseic met˘ jeshc

$$\widehat{\varphi}(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\frac{\widehat{\alpha}_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}} e^{i\vec{x}\cdot\vec{p}} + \frac{\widehat{a}^{\dagger}_{\vec{p}}}{\sqrt{2\omega_{\vec{p}}}} e^{-i\vec{x}\cdot\vec{p}} \right) \tag{C.5.7}$$

$$\widehat{pi}(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_{\vec{p}}}{2}} \right) \left(\widehat{\alpha}_{\vec{p}} e^{i\vec{x}\cdot\vec{p}} - \widehat{\alpha}^{\dagger}_{\vec{p}} e^{-i\vec{x}\cdot\vec{p}} \right) \tag{C.5.8}$$

$$[\widehat{\alpha}_{\vec{p}}, \widehat{\alpha}^{\dagger}_{\vec{p}\,'}] = (2\pi)^3\delta^3(\vec{p}-\vec{p}\,') \tag{C.5.9}$$

$$[\widehat{\alpha}_{\vec{p}}, \widehat{\alpha}_{\vec{p}\,'}] = [\widehat{\alpha}^{\dagger}_{\vec{p}}, \widehat{\alpha}^{\dagger}_{\vec{p}\,'}] = 0 \tag{C.5.10}$$

Oi exis¸seic C.5.9 kai C.5.10 prokÔptoun apì tic exis¸seic C.5.6-C.5.8.

EpÐshc prowjoÔntai se telestèc h qamiltonian $H \to \widehat{H}$ kai h qamiltonian puknìthta $\mathcal{H} \to \widehat{\mathcal{H}}$, ètsi ¸ste $\widehat{H} = \int \widehat{\mathcal{H}}\, d^3\vec{x}$. H qamiltonian paÐrnei thn morf

$$\widehat{H} = \int d^3\vec{x} \left(\frac{1}{2}\widehat{\pi}^2(\vec{x}) + \frac{1}{2}\left(\nabla\widehat{\varphi}(\vec{x}) \right)^2 + \frac{1}{2}m^2\widehat{\varphi}^2(\vec{x}) \right) \tag{C.5.11}$$

kai sunart sei twn telest¸n dhmiourgÐac kai katastrof c

$$\widehat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(\omega_{\vec{p}}\widehat{\alpha}^{\dagger}_{\vec{p}}\widehat{\alpha}_{\vec{p}} + \omega_{\vec{p}}[\widehat{\alpha}_{\vec{p}}, \widehat{\alpha}^{\dagger}_{\vec{p}}] \right) \tag{C.5.12}$$

O deÔteroc ìroc sto olokl rwma apeirÐzetai afoÔ eÐnai an˘logoc thc posìthtac $(2\pi)^3\delta^3(0)$. Se poll˘ peir˘mata metroÔme mìno diaforèc enèrgeiac. Epomènwc autìc o ìroc den suneisfèrei – antistoiqeÐ sthn enèrgeia tou kenoÔ. Gr˘fontac

$$(2\pi)^3\delta^3(\vec{p}-\vec{p}\,') = \int d^3\vec{x} e^{i(\vec{p}-\vec{p}\,').\vec{x}} \Rightarrow (2\pi)^3\delta^3(0) = \int d^3\vec{x} = V$$

paÐrnoume thn èkfrash gia thn enèrgeia thc jemeli¸douc kat^stashc (kat^stashc kenoÔ)

$$E_0 = V \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\omega_{\vec{p}}}{2} \tag{C.5.13}$$

Oi metajètec metaxÔ thc qamiltonian c kai twn telest¸n dhmiourgÐac kai katastrof c eÐnai

$$[\widehat{H}, \widehat{\alpha}^\dagger_{\vec{p}}] = \omega_{\vec{p}} \widehat{\alpha}^\dagger_{\vec{p}} \quad \kappa\alpha\iota \quad [\widehat{H}, \widehat{\alpha}_{\vec{p}}] = -\omega_{\vec{p}} \widehat{\alpha}_{\vec{p}} \tag{C.5.14}$$

Gia na apaleif jeÐ o ìroc pou apeirÐzetai apì thn exÐswsh C.5.12 orÐzoume wc nèa qamiltonian thn $\widehat{H}' = \widehat{H} - E_0$, h opoÐa èqei thn akìloujh morf

$$\widehat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} \widehat{\alpha}^\dagger_{\vec{p}} \widehat{\alpha}_{\vec{p}} \tag{C.5.15}$$

H kat^stash kenoÔ $|0\rangle$ eÐnai h kat^stash sthn opoÐa ìtan dr^sei o telest c katastrof c $\widehat{\alpha}_{\vec{p}}$ dÐnei mhdèn, $\widehat{\alpha}_{\vec{p}}|0\rangle = 0$, gia ìlec ti timèc tou $\vec{p}$. An dr^sei o telest c $\widehat{H}$ sthn kat^stash $\widehat{\alpha}^\dagger_{\vec{p}}|0\rangle$ paÐrnoume

$$\widehat{H}\widehat{\alpha}^\dagger_{\vec{p}}|0\rangle = \omega_{\vec{p}} \widehat{\alpha}^\dagger_{\vec{p}}|0\rangle \tag{C.5.16}$$

ParathroÔme ìti h pio p^nw kat^stash apoteleÐ idiokat^stash thc qamiltonian c, me idiotim Ðsh me $\omega_{\vec{p}}$. O telest c dhmiourgÐac dhmiourgeÐ èna swmatÐdio.

O telest c thc orm c dÐnetai apì thn sqèsh

$$\vec{P} = -\int d^3\vec{x} \pi(x) \vec{\nabla}\varphi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} \widehat{\alpha}^\dagger_{\vec{p}} \widehat{\alpha}_{\vec{p}} \tag{C.5.17}$$

K^je kat^stash me kajorismènh enèrgeia kai orm prokÔptei apì thn dr^sh telest¸n dhmiourgÐac sthn kat^stash kenoÔ. K^je dr^sh telest dhmiourgÐac se mia kat^stash prosjètei enèrgeia $\omega_{\vec{p}}$ kai orm $\vec{p}$.

Epilègoume thn kat^stash kenoÔ na eÐnai kanonikopoihmènh $\langle 0|0\rangle = 1$. H kat^stash me èna swmatÐdio kai orm $\vec{p}$ eÐnai h $\widehat{\alpha}^\dagger_{\vec{p}}|0\rangle$. Oi katast^seic eÐnai orjog¸niec metaxÔ touc: $\langle 0|\widehat{\alpha}_{\vec{q}} \widehat{\alpha}^\dagger_{\vec{p}}|0\rangle = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$. 'Omwc to eswterikì ginìmeno metaxÔ dÔo katast^sewn prèpei na paramènei amet^blhto k^tw apì metasqhmatismoÔc Lorentz. Gia na ikanopoihjeÐ to aÐthma autì orÐzoume thn kat^stash enìc swmatidÐou wc

$$\sqrt{2E_{\vec{p}}} \widehat{\alpha}^\dagger_{\vec{p}}|0\rangle = |\vec{p}\rangle \tag{C.5.18}$$

C.5.3 Diadìthc Klein-Gordon

Arqik˘ orÐzoume thn posìthta

$$D_R \equiv \Theta(x^0 - y^0)\langle 0|[\varphi(x), \varphi(y)]|0\rangle \tag{C.5.19}$$

E˘n dr˘soume se aut thn posìthta me ton telest Klein-Gordon brÐskoume

$$(\partial^\mu \partial_\mu + m^2)D_R \equiv \Theta(x^0 - y^0)\langle 0|[\varphi(x), \varphi(y)]|0\rangle = -i\delta^3(\vec{x} - \vec{y}) \tag{C.5.20}$$

Sunep¸c h D_R apoteleÐ mia sun˘rthsh Green thc exÐswshc Klein-Gordon. MetasqhmatÐzontac kat˘ Fourier mporoÔme na broÔme mia èkfrash gia thn D_R

$$D_R(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip(x-y)} \tag{C.5.21}$$

ParathroÔme ìti to olokl rwma wc proc p^0 èqei 2 pìlouc sta shmeÐa $p^0 = \pm E_{\vec{p}}$. SÔmfwna me to je¸rhma twn Cauchy - Goursat [30] up˘rqoun 4 diaforetikoÐ trìpoi na upologisteÐ to olokl rwma, an˘loga me to e˘n oi pìloi metakinhjoÔn apì ton pragmatikì ˘xona, ¸ste na apokt soun jetik   arnhtik fantastik sunist¸sa. H k˘je epilog antistoiqeÐ se mia diaforetik sun˘rthsh Green. Gia na p˘roume ton diadìth Feynman prosdÐdoume ston arnhtikì pìlo jetik fantastik sunist¸sa kai ston jetikì pìlo arnhtik me ton ex c trìpo

$$D_R(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip.(x-y)} \tag{C.5.22}$$

'Etsi h èkfrash gia ton diadìth Feynman paÐrnei thn morf

$$D_F(x - y) = \Theta(x^0 - y^0)\langle 0|\varphi(x)\varphi(y)|0\rangle + \Theta(y^0 - x^0)\langle 0|\varphi(y)\varphi(x)|0\rangle \tag{C.5.23}$$

Qrhsimopoi¸ntac to sÔmbolo thc qronik c seir˘c h èkfrash C.5.23, gr˘fetai sthn morf

$$D_F(x - y) = \langle 0|T\{\varphi(x)\varphi(y)\}|0\rangle \tag{C.5.24}$$

Ja qrhsimopoi soume to apotèlesma autì gia na broÔme touc kanìnec Feynman gia bajmwt˘ pedÐa.

C.6 PedÐo Dirac

C.6.1 Klasik exÐswsh Dirac

H exÐswsh Dirac èqei thn morf

$$(i\partial\!\!\!/ - m)\Psi = 0 \tag{C.6.1}$$

ìpou $\partial\!\!\!/ = \partial_\mu \gamma^\mu$. Se antÐjesh me to pedÐo Klein-Gordon, to pedÐo Ψ eÐnai èna spinoriakì pedÐo me 4 migadikèc sunist¸sec. AntistoiqeÐ se swmatÐdia kai antiswmatÐdia me spin $1/2$ (fermiìnia). Tètoia swmatÐdia eÐnai ta hlektrìnia kai ta pozitrìnia. Oi pÐnakec γ^μ eÐnai tetragwnikoÐ pÐnakec 4×4. Sthn qeiralik b˘sh èqoun thn morf

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \bar{\sigma}^\mu = (1, -\sigma^i), \quad \sigma^\mu = (1, \sigma^i) \tag{C.6.2}$$

ìpou σ^i eÐnai oi pÐnakec Pauli.

H lagkrazian tou pedÐou Dirac eÐnai

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi \tag{C.6.3}$$

ìpou $\bar{\Psi} = \Psi^\dagger \gamma^0$. Oi spÐnorec Dirac metasqhmatÐzontai k˘tw apì metasqhmatismoÔc Lorentz me ton akìloujo trìpo [11]

$$\Psi' = \Lambda_{\frac{1}{2}} \Psi \tag{C.6.4}$$

ìpou $\Lambda_{\frac{1}{2}}$ eÐnai ènac 4×4 pÐnakac me thn ex c idiìthta [11]

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu \gamma^\nu \tag{C.6.5}$$

ìpou o $\Lambda^\mu{}_\nu$ eÐnai o pÐnakac pou metasqhmatÐzei tetradianÔsmata. Qrhsimopoi¸ntac aut thn idiìthta blèpoume amèswc ìti h exÐswsh Dirac kai h lagkrazian mènoun amet˘blhtec k˘tw apì metasqhmatismoÔc Lorentz.

H suzug c orm dÐnetai apì thn èkfrash

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\Psi}} = i\Psi^\dagger \tag{C.6.6}$$

kai h qamiltonian tou pedÐou Dirac

$$H = \int d^3\vec{x} \left(\frac{\partial \mathcal{L}}{\partial \dot{\Psi}} \dot{\Psi} - \mathcal{L} \right) = \int d^3\vec{x} \Psi^\dagger \left(-i\gamma^0 \vec{\gamma} \cdot \vec{\nabla} + \gamma^0 m \right) \Psi \tag{C.6.7}$$

C.6.2 Kb^ntwsh pedÐou Dirac

Kbant¸noume to pedÐo Ψ an^gontac to se telest , ìpwc kai thn suzug tou orm , kai epib^loume sqèseic antimet^jeshc [3]. Oi sqèseic antimet^jeshc (th qronik stigm $t = 0$) eÐnai

$$\{\Psi_\alpha(\vec{x}), \Psi^\dagger_\beta(\vec{y})\} = \delta^3(\vec{x} - \vec{y})\delta_{\alpha\beta}, \quad \{\Psi^\dagger_\alpha(\vec{x}), \Psi^\dagger_\beta(\vec{y})\} = \{\Psi_\alpha(\vec{x}), \Psi_\beta(\vec{y})\} = 0 \tag{C.6.8}$$

To pedÐo Ψ mporeÐ na graf teÐ sthn morf

$$\Psi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(\alpha^s_{\vec{p}} u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^s_{\vec{p}} v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right) \tag{C.6.9}$$

O $\alpha^s_{\vec{p}}$ eÐnai o telest c katastrof c kai o $\alpha^{s\dagger}_{\vec{p}}$ o telest c dhmiourgÐac, ìpwc sthn perÐptwsh tou bajmwtoÔ pedÐou Klein-Gordon. Oi telestèc $b^s_{\vec{p}}$ kai $b^{s\dagger}_{\vec{p}}$ eÐnai oi telestèc katastrof c kai dhmiourgÐac antiswmatidÐwn. O deÐkthc s anafèretai stic 2 anex^rthtec sunist¸sec pìlwshc twn swmatidÐwn, exaitÐac tou mh mhdenikoÔ touc spin.

Oi spÐnorec u, v apoteloÔn lÔseic thc exÐswshc Dirac gia jetik kai arnhtik suqnìthta antÐstoiqa

$$\Psi(x) = u(\vec{p})e^{-ipx}, \quad u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix}, \quad p^0 > 0 \tag{C.6.10}$$

$$\Psi(x) = v(\vec{p})e^{ipx}, \quad v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}, \quad p^0 > 0 \tag{C.6.11}$$

Oi spÐnorec ξ kai η èqoun 2 sunist¸sec kai ikanopoioÔn tic sqèseic

$$\sum_{s=1,2} \xi^s\xi^{s\dagger} = \sum_{s=1,2} \eta^s\eta^{s\dagger} = 1, \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{C.6.12}$$

H sunj kh kanonikopoÐhshc pou epib^lloume eÐnai

$$\bar{u}u = 2m\xi^\dagger\xi, \quad \bar{v}v = 2m\eta^\dagger\eta, \quad \bar{u} = u^\dagger\gamma^0 \quad \bar{v} = v^\dagger\gamma^0 \tag{C.6.13}$$

kai h sunj kec orjogwniìthtac eÐnai

$$\bar{u}^r(\vec{p})u^s(\vec{p}) = 2m\delta^{rs}, \quad \bar{v}^r(\vec{p})v^s(\vec{p}) = 2m\delta^{rs}, \quad \bar{u}^r(\vec{p})v^s(\vec{p}) = -2m\delta^{rs} \tag{C.6.14}$$

[3] Στην περίπτωση του πεδίου Dirac προκύπτει πρέπει να επιβάλουμε σχέσεις αντιμετάθεσης, ενώ στην περίπτωση βαθμωτών πεδίων επιβάλουμε σχέσεις μετάθεσης [11].

EpÐshc isqÔoun oi sqèseic

$$\sum_s u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m, \quad \sum_s v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m \tag{C.6.15}$$

Apì tic sqèseic C.6.8 kai C.6.9, brÐskoume tic sqèseic antimetˆjeshc

$$\{\alpha^s_{\vec{p}}, \alpha^{r\dagger}_{\vec{q}}\} = \{b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\} = (2\pi)^3\delta^3(\vec{p} - \vec{q})\delta^{sr} \tag{C.6.16}$$

$$\{\alpha^s_{\vec{p}}, \alpha^r_{\vec{q}}\} = \{b^s_{\vec{p}}, b^r_{\vec{q}}\} = \{\alpha^{s\dagger}_{\vec{p}}, \alpha^{r\dagger}_{\vec{q}}\} = \{b^{s\dagger}_{\vec{p}}, b^{r\dagger}_{\vec{q}}\} = 0 \tag{C.6.17}$$

Oi katastˆseic dÔo (perissìterwn) swmatidÐwn eÐnai antisummetrikèc

$$\alpha^{s\dagger}_{\vec{p}}\alpha^{s\dagger}_{\vec{q}}|0\rangle = -\alpha^{s\dagger}_{\vec{q}}\alpha^{s\dagger}_{\vec{p}}|0\rangle \tag{C.6.18}$$

kai jètontac $\vec{p} = \vec{q}$ èqoume

$$(\alpha^{s\dagger}_{\vec{p}})^2|0\rangle = 0 \tag{C.6.19}$$

H pio pˆnw sqèsh eÐnai gnwst wc h apagoreutik arq tou Pauli.

H qamiltonian paÐrnei thn morf

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3}\sum_s E_{\vec{p}}\left(\alpha^{s\dagger}_{\vec{p}}\alpha^s_{\vec{p}} + b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}\right) \tag{C.6.20}$$

kai o telest c thc orm c gÐnetai

$$\vec{P} = \int d^3\vec{x}\,\Psi^\dagger(-i\nabla)\Psi = \int \frac{d^3\vec{p}}{(2\pi)^3}\sum_s \vec{p}\left(\alpha^{s\dagger}_{\vec{p}}\alpha^s_{\vec{p}} + b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}\right) \tag{C.6.21}$$

Oi monoswmatidiakèc katastˆseic eÐnai

$$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\,\alpha^{s\dagger}_{\vec{p}}|0\rangle \tag{C.6.22}$$

kai h sunj kh orjokanonikìthtac touc eÐnai

$$\langle\vec{p}, r|\vec{q}, s\rangle = 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p} - \vec{q})\delta^{rs} \tag{C.6.23}$$

To diˆnusma

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{C.6.24}$$

ikanopoieÐ thn exÐswsh sunèqeiac, $\partial_\mu j^\mu = 0$. H qronik sunist¸sa dÐnei k˜poio diathroÔmeno fortÐo, to opoÐo isoÔtai me

$$Q = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_s \left(\alpha_{\vec{p}}^{s\dagger} \alpha_{\vec{p}}^{s} - b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s} \right) \tag{C.6.25}$$

Sthn kbantik hlektrodunamik o telest c autìc (epÐ to fortÐo tou hlektronÐou) eÐnai o telest c tou hlektrikoÔ fortÐou.

C.6.3 Diadìthc Feynman gia to pedÐo Dirac

Gia na orÐsoume to diadìth Feynman, ja qreiastoÔme tic posìthtec

$$\langle 0|\Psi_a(x)\bar{\Psi}_b(y)|0\rangle = (i\not{\partial}_x + m)_{ab} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \tag{C.6.26}$$

$$\langle 0|\bar{\Psi}_b(y)\Psi_a(x)|0\rangle = -(i\not{\partial}_x + m)_{ab} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(y-x)} \tag{C.6.27}$$

H sun˜rthsh Green thc exÐswshc Dirac

$$S_R^{ab} \equiv \Theta(x^0 - y^0)\langle 0|\{\Psi_a(x), \bar{\Psi}_b(y)\}|0\rangle \tag{C.6.28}$$

ikanopoieÐ thn sqèsh

$$(i\not{\partial}_x - m)S_R = i\delta^4(x-y) \cdot 1_{4\times 4} \tag{C.6.29}$$

MetasqhmatÐzontac kat˜ Fourier brÐskoume

$$S_R(p) = \frac{i(\not{p} + m)}{p^2 - m^2} \tag{C.6.30}$$

O diadìthc Feynman paÐrnei thn morf

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} = \langle 0|T\Psi(x)\bar{\Psi}(y)|0\rangle \tag{C.6.31}$$

ìpou to sÔmbolo T eÐnai to telest c qronik c seir˜c me thn sÔmbash ìti h èkfrash pollaplasi˜zetai me èna par˜gonta (-1) gia k˜je enallag thc seir˜c twn pedÐwn.

C.7 Kanìnec Feynman gia thn kbantik hlektrodunamik

Oi jewrÐec Dirac kai Klein-Gordon eÐnai eleÔjerec jewrÐec me th lagkrazian pou tic perigr˜fei na mhn èqei ìrouc allhlepÐdrashc. Oi ìroi allhlepÐdrashc miac lagkrazian c mporeÐ na perilambˆnoun ginìmena bajmwt¸n pedÐwn, pedÐwn bajmÐdac (ìpwc to dianusmatikì dunamikì A_μ), spinìrwn k.t.l. Oi ìroi autoÐ prèpei na eÐnai ametˆblhtoi kˆtw apì metasqhmatismoÔc Lorentz. EpÐshc h jewrÐa prèpei na eÐnai epanakanonikopoi shmh [11], èna aÐthma pou epitrèpei mìno èna peperasmèno arijmì ìrwn. (P.q. oi diastˆseic twn suntelest¸n touc den prèpei na eÐnai dunˆmeic m kouc).

AkoloÔjwc exetˆzoume thn lagkrazian thc kbantik c hlektrodunamik c. H jewrÐa aut perilambˆnei fermiìnia Dirac, to hlektromagnhtikì dunamikì A_μ kai perigrˆfei tic allhlepidrˆseic metaxÔ touc.

C.7.1 QED

H lagkrazian thc QED dÐnetai apì thn èkfrash

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + q\bar{\Psi}\gamma^\mu\Psi A_\mu \tag{C.7.1}$$

ìpou $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ o tanust c tou hlektromagnhtikoÔ pedÐou kai $q = -e$ to fortÐo tou hlektronÐou. MporoÔme na sumperilˆboume perissìtera fermionikˆ pedÐa me parìmoiec allhlepidrˆseic.

H lagkrazian grˆfetai sthn morf

$$\mathcal{L}_{QED} = \bar{\Psi}(iD\!\!\!\!/ - m)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{C.7.2}$$

ìpou D_μ h sunalloÐwth parˆgwgoc

$$D_\mu \equiv \partial_\mu + ieA_\mu(x) \tag{C.7.3}$$

H sunalloÐwth parˆgwgoc metasqhmatÐzetai wc èna sunalloÐwto tetradiˆnusma kˆtw apì metasqhmatismoÔc Lorentz. H lagkrazian paramènei ametˆblhth kˆtw apì metasqhmatismoÔc bajmÐdac

$$\Psi'(x) = e^{i\alpha(x)}\Psi(x), \quad A'_\mu(x) = A_\mu - \frac{1}{e}\partial_\mu\alpha(x) \tag{C.7.4}$$

To diathroÔmeno fortÐo Noether pou prokÔptei apì aut n thn summetrÐa eÐnai to hlektrikì fortÐo sthn perÐptwsh pou h bajmÐda eÐnai h bajmwt sunˆrthsh.

H exÐswsh kÐnhshc tou $\Psi(x)$ dÐnetai apì thn èkfrash

$$(iD\!\!\!\!/ - m)\Psi(x) = 0 \tag{C.7.5}$$

kai h exÐswsh kÐnhshc tou dunamikoÔ A_μ apì thn

$$\partial_\mu F^{\mu\nu} = e\bar{\Psi}\gamma^\nu\Psi \tag{C.7.6}$$

Jètontac $\bar{\Psi}\gamma^\nu\Psi = j^\nu$, ìpou j^ν h puknìthta tou hlektrikoÔ reÔmatoc, paÐrnoume tic exis¸seic Maxwell.

H kb˘ntwsh thc QED epitugq˘netai pio eÔkola me thn qr sh sunarthsiak¸n oloklhrwm˘twn. Gia suntomÐa ja parousi˘soume touc kanìnec Feynman.

C.7.2 PÐnakac skèdashc S

'Estw ìti èqoume mia arqik kat˘stash dÔo swmatidÐwn $|\mathrm{k}_A\mathrm{k}_B\rangle_{in}$ me ormèc $\vec{\mathrm{k}}_A, \vec{\mathrm{k}}_B$. To pl˘toc pijanìthtac aut h kat˘stash na katal xei se mia telik kat˘stash $|\mathrm{p}_1\mathrm{p}_2\ldots\rangle$ dÐnetai apì thn èkfrash

$$ {}_{out}\langle \mathrm{p}_1\mathrm{p}_2\ldots|\mathrm{k}_A\mathrm{k}_B\rangle_{in} \tag{C.7.7} $$

H arqik kat˘stash orÐzetai se qrìno $-T$ sto pareljìn, ìpou T teÐnei sto ˘peiro. AntÐstoiqa h telik kat˘stash orÐzetai se qrìno $+T$. Epomènwc to pl˘toc skèdashc C.7.7 paÐrnei thn morf

$$ {}_{out}\langle \mathrm{p}_1\mathrm{p}_2\ldots|\mathrm{k}_A\mathrm{k}_B\rangle_{in} = \lim_{T\to\infty}\langle \mathrm{p}_1\mathrm{p}_2\ldots|e^{-2iTH}|\mathrm{k}_A\mathrm{k}_B\rangle \tag{C.7.8} $$

Sto ìrio ìpou o qrìnoc T teÐnei sto ˘peiro, o telest c exèlixhc e^{-2iTH} an˘getai ston pÐnaka skèdashc S. O pÐnakac S eÐnai monadiakìc: $S^\dagger S = 1$.

E˘n ta dÔo arqik˘ swmatÐdia den allhlepidroÔn o pÐnakac S tautÐzetai me ton monadiaÐo telest . Se jewrÐec me mh tetrimmènec allhlepidr˘seic, o pÐnakac skèdashc gr˘fetai sthn morf

$$ S = 1 + iT \tag{C.7.9} $$

ìpou T eÐnai to mèroc pou perigr˘fei tic allhlepidr˘seic.

AfoÔ h orm diathreÐtai oi pÐnakec S kai T prèpei na perièqoun èna par˘gonta $\delta^4\big(k_A + k_B - \sum_f p_f\big)$. Ex˘gontac autìn ton par˘gonta, orÐzoume to amet˘blhto pl˘toc $\mathcal{M}$

$$ \langle \mathrm{p}_1\mathrm{p}_2\ldots|iT|\mathrm{k}_A\mathrm{k}_B\rangle = (2\pi)^4\delta^4\big(k_A + \kappa_B - \sum_f p_f\big)i\mathcal{M}(k_A, k_B \to p_f) \tag{C.7.10} $$

H pijanìthta gia th diergasÐa C.7.7 dÐnetai apì thn èkfrash

$$ \mathcal{P}(A, B \to 1\ 2\ldots n) = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{E_f}\right) |{}_{out}\langle \mathrm{p}_1\mathrm{p}_2\ldots|\mathrm{k}_A\mathrm{k}_B\rangle_{in}|^2 \tag{C.7.11} $$

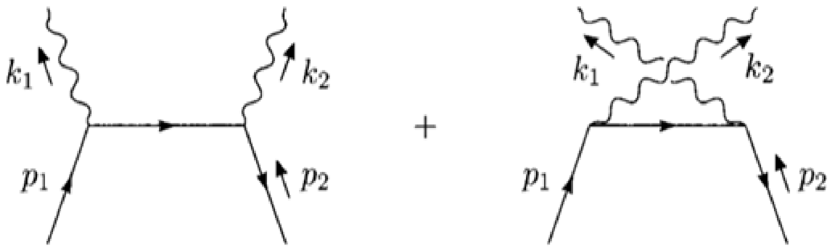
H diaforik diatom skèdashc apì dÔo se dÔo swmatÐdia dÐdetai apì thn èkfrash [11]

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{|\vec{p}_1|}{(2\pi)^2 4E_{cm}}|\mathcal{M}(p_A, p_B \to p_1, p_2|^2 \tag{C.7.12}$$

ìpou $v_A - v_B$ eÐnai h sqetik taqÔthta metaxÔ twn arqik¸n swmatidÐwn.

C.7.3 Upologismìc tou pÐnaka skèdashc apì diagr^mmata Feynman

'Estw ìti èqoume mia skèdash $e^- e^+ \to \gamma\gamma$. To pl^toc $i\mathcal{M}$ dÐnetai apì to ^jroisma twn akìloujwn diagramm^twn Feynman



EIKONA C.7.1: Diagr^mmata Feynman gia ton upologismì tou pl^touc $i\mathcal{M}$ gia thn skèdash $e^- e^+ \to \gamma\gamma$ [11]. To pinakostoiqeÐo dÐnetai apì to ^jroisma 2 diagramm^twn afoÔ ta telik^ swmatÐdia eÐnai panomoiìtupa kai den mporoÔme na diakrÐnoume poio èqei orm k_1 kai poio k_2.

Ta fermiìnia sumbolÐzontai me suneqìmenec grammèc kai ta fwtìnia me kumatistèc grammèc. Oi grammèc twn antifermionÐwn èqoun bèloc antÐjeto me to bèloc thc orm c. K^je shmeÐo pou sunant¸ntai treic grammèc (2 fermiìnia kai 1 fwtìnio) onom^zetai koruf kai proèrqetai apì ton antÐstoiqo ìro allhlepÐdrashc sthn lagkrazian . Oi grammèc pou èqoun mia eleÔjerh ^krh onom^zontai exwterikèc, en¸ autèc pou kai oi dÔo ^krec touc katal goun se korufèc onom^zontai eswterikèc. Oi eswterikèc grammèc antistoiqoÔn stouc diadìtec. Se k^je koruf prèpei na diathreÐtai h orm . Sta pio p^nw diagr^mmata to bèloc tou qrìnou eÐnai proc ta p^nw. Sto pr¸to di^gramma èqoume sthn Ðdia koruf tic ormèc p_1, k_1 kai p_2, k_2 antÐstoiqa, en¸ sto deÔtero di^gramma h orm p_1 sundèetai me thn orm k_2.

Gia na broÔme thn èkfrash gia to pl^toc $i\mathcal{M}$, akoloujoÔme touc akìloujouc kanìnec Feynman:

1) Diadìtec

F Diadìthc fermionÐou

$$\frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \tag{C.7.13}$$

F Diadìthc fwtonÐou

$$\frac{ig_{\mu\nu}}{q^2 + i\epsilon} \tag{C.7.14}$$

ìpou μ, ν oi sunist¸sec twn dianusm˘twn pìlwshc twn fwtonÐwn stic 2 ˘krec tou diadìth.

2) Korufèc
F Koruf me 2 fermiìnia kai èna fwtìnio

$$-ie\gamma^{\mu} \tag{C.7.15}$$

3) Exwterikèc grammèc
F Eiserqìmeno kai exerqìmeno fermiìnio

$$u^s(p), \quad \bar{u}^s(p) \tag{C.7.16}$$

F Eiserqìmeno kai exerqìmeno antifermiìnio

$$\bar{v}^s(p), \quad v^s(p) \tag{C.7.17}$$

F Eiserqìmeno kai exerqìmeno fwtìnio

$$\vec{\epsilon}^{\mu}(p), \quad \epsilon^{*\mu}(p) \tag{C.7.18}$$

4) Diat rhsh orm c se k˘je koruf
5) Olokl rwsh wc proc ìlec tic ormèc brìqou
6) EÔresh tou sunolikoÔ pros mou tou diagr˘mmatoc
7) DiaÐresh me ton par˘gonta summetrÐac tou diagr˘mmatoc

Gia na broÔme thn èkfrash gia to pl˘toc $i\mathcal{M}$ prèpei na pollaplasi˘soume touc par˘-gontec pou prokÔptoun apì touc kanìnec 1-4,7 gia k˘je di˘gramma xeqwrist˘, kai akoloÔjwc na prosjèsoume ìla ta diagr˘mmata pou suneisfèroun. AfoÔ to k˘noume autì oloklhr¸-noume wc proc ìlec tic ormèc brìqwn pou apomènoun.

Parˆrthma D

PÐnakec puknìthtac

'Enac pÐnakac puknìthtac $\widehat{\rho}$ èqei tic exˆc treic idiìthtec:

1) $\widehat{\rho}^{\dagger} = \widehat{\rho}$
2) $\mathrm{Tr}\,\widehat{\rho} = 1$
3) Oi idiotimèc tou λ_i eÐnai mh arnhtikèc mikrìterec Ðsec me 1.

'Estw ìti èqoume èna kbantikì sÔsthma A. Oi katastˆseic $|\alpha_i\rangle$ eÐnai idiokatastˆseic enìc telest $\widehat{A}$ me idiotimèc α_i. Oi katastˆseic autèc apoteloÔn èna pl rec orjokanonikì sÔnolo. Eˆn o pÐnakac puknìthtac pou perigrˆfei to sÔsthma eÐnai diag¸nioc sth bˆsh aut , ja dÐnetai apì thn èkfrash:

$$\widehat{\rho} = \sum_i \lambda_i \, |\alpha_i\rangle \langle\alpha_i| \tag{D.1}$$

Eˆn to sÔsthma A brÐsketai se mia kajar katˆstash, $\mathrm{Tr}\,\rho^2 = 1$ en¸ se mia meikt katˆstash $\mathrm{Tr}\,\rho^2 < 1$.

'Estw ìti èqoume èna sÔsthma pou apoteleÐtai apì ta uposust mata A kai B. Genikˆ ta uposust mata ja brÐskontai se kbantik sÔmplexh. 'Estw to sunolikì sÔsthma perigrˆfetai apì mia kajar katˆstash Ψ. IsodÔnama to sunolikì sÔsthma perigrˆfetai apì ton pÐnaka puknìthtac $\widehat{\rho}_{AB} = |\Psi\rangle\langle\Psi|$. To uposÔsthma A perigrˆfetai apì ton pÐnaka puknìthtac $\widehat{\rho}_A = \mathrm{Tr}_B\,|\Psi\rangle\langle\Psi|$, ìpou to Tr_B to merikì Ðqnoc wc proc to uposÔsthma B. Oi pÐnakec puknìthtac ρ_A kai ρ_B èqoun Ðsec mh mhdenikèc idiotimèc. Oi telestèc pou ephreˆzoun mìno to uposÔsthma A èqoun thn morf : $\widehat{Q}_A \otimes \widehat{1}_B$. Tìte

$$\langle\Psi|\, \widehat{Q}_A \otimes \widehat{1}_B \,|\Psi\rangle = \mathrm{Tr}\Big\{\widehat{\rho}_A \widehat{Q}_A\Big\} \tag{D.2}$$

H kanonik sullog perigr˘fetai èna jermikì pÐnaka puknìthtac

$$\widehat{\rho} = \sum_i \frac{1}{Z} \left| E_i \right\rangle \left\langle E_i \right| e^{-\frac{E_i}{T}} \tag{D.3}$$

ìpou T h jermokrasÐa kai $\lambda_i = e^{-E_i/T}/Z$ eÐnai h pijanìthta to sÔsthma na brÐsketai se mia kat˘stash me enèrgeia E_i. H kat˘stash $|E_i\rangle$ eÐnai idiokat˘stash thc qamiltonian c me idiotim E_i. QrhsimopoioÔme mon˘dec stic opoÐec h stajer˘ Boltzman kai h stajer˘ tou Planck ~ isoÔntai me mon˘da.

Par^rthma E

Om^dec sthn fusik

Sthn fusik sunant^me suqn^ kathgorÐec metasqhmatism¸n ìpwc oi peristrofèc, oi metatopÐseic jèshc, oi metasqhmatismoÐ Lorentz k.t.l. Ta stoiqeÐa thc k^je kathgorÐac metasqhmatism¸n parousi^zoun di^forec idiìthtec tic opoÐec mporoÔme eÔkola na melet - soume eis^gontac thn ènnoia thc om^dac.

E.1 OrismoÐ

Om^da eÐnai èna sÔnolo stoiqeÐwn G me mia pr^xh $\circ$, ètsi ¸ste $\forall g_i, g_j \in G, \;\; g_i \circ g_j \in G$. Prèpei na isqÔoun oi ex c idiìthtec:

1)$(g_i \circ g_j) \circ g_k = g_i \circ (g_j \circ g_k)$

2)$\exists e$ t.w $e \circ g_i = g_i \circ e = g_i, \quad \forall g_i \in G$

3)$\forall g_i \in G, \; \exists g_i^{-1}$ t.w $g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = e$

To stoiqeÐo e eÐnai to monadiaÐo stoiqeÐo kai suqn^ ja to sumbolÐzoume me 1.

An gia mia om^da isqÔei $g_i \circ g_j = g_j \circ g_i, \; \forall g_i, g_j \in G$, tìte h om^da eÐnai abelian alli¸c h om^da eÐnai mh abelian .

Pollèc om^dec èqoun ^lgebrec. Thn ^lgebra thc om^dac ja antiproswpeÔoun ta stoiqeÐa T^i, ta opoÐa onom^zontai genn torec kai sundèontai me ta stoiqeÐa g_i thc om^dac mèsw thc sqèshc $g_i = e^{i\alpha_j T^j}$. Gia mia ^lgebra isqÔei h sqèsh $[T^i, T^j] = i f^{ijk} T^k$, ìpou oi arijmoÐ f^{ijk} onom^zontai stajerèc dom c.

E.2 Om^da peristrof¸n

Ac melet soume pr¸ta thn om^da twn peristrof¸n. 'Estw ìti peristrèfoume èna n-di^stato di^nusma $\vec{a}$ ston eukleÐdeio q¸ro. Gia na to k^noume autì qrhsimopoioÔme èna pragmatikì pÐnaka peristrof c $n \times n$

$$\vec{a}' = R\vec{a} \tag{E.2.1}$$

Stic peristrofèc to mètro tou dianÔsmatoc paramènei stajerì. Epomènwc

$$\vec{a}'^T \vec{a}' = \vec{a}^T \vec{a} = \vec{a}^T R^T R \vec{a} \tag{E.2.2}$$

Katal goume sthn sqèsh gia touc pÐnakec peristrof c

$$R^T R = 1 \tag{E.2.3}$$

O pÐnakac R gia na apoteleÐ pÐnaka peristrof c prèpei na eÐnai ènac orjog¸nioc pÐnakac (dhlad upakoÔei sthn exÐswsh E.2.3). T¸ra jèloume na exet˘soume e˘n oi peristrofèc apoteloÔn om˘da. H pr˘xh eÐnai o pollaplasiasmìc pin˘kwn. Katarq˘c e˘n efarmìsoume 2 peristrofèc, thn mia met˘ thn ˘llh, to apotèlesma ja eÐnai kai p˘li mia peristrof . H idiìthta 1 isqÔei gia ìlouc touc pÐnakec kai epomènwc isqÔei kai gia touc orjog¸niouc pÐnakec. EpÐshc up˘rqei to monadiaÐo stoiqeÐo pou eÐnai o monadiaÐoc pÐnakac, o opoÐoc ìtan dr˘sei se èna di˘nusma to af nei analloÐwto. Epomènwc kai h deÔterh sunj kh ikanopoieÐtai. 'Oso gia thn trÐth sunj kh mporoÔme p˘nta se mia peristrof na efarmìsoume antÐjeth peristrof kai to di˘nusma na epanèljei sthn arqik tou kat˘stash. 'Etsi ikanopoieÐtai kai h trÐth sunj kh. Epomènwc oi orjog¸nioi pÐnakec apoteloÔn om˘da. Se tuqaÐouc pÐnakec peristrof c genik˘ den isqÔei h antimetajetik idiìthta. Epomènwc h om˘da eÐnai mh abelian .

H ˘lgebra thn om˘dac twn orjog¸niwn pin˘kwn eÐnai oi antisummetrikoÐ pÐnakec $n \times n$ ($T^T = -T$). Genik˘ gia n dias t˘seic qrei˘zontai $\frac{n(n-1)}{2}$ par˘metroi gia na prosdioristeÐ mia peristrof , ìpwc faÐnetai apì thn sqèsh E.2.3, pio xek˘jara apì thc sqèsh antisummetrikìthtac tou genn tora. H om˘da twn n-di˘statwn orjog¸niwn pin˘kwn sumbolÐzetai me $O(n)$. H om˘da twn orjog¸niwn pin˘kwn perilamb˘nei peristrofèc kai katoptrismoÔc. Gia na perioristoÔme stic peristrofèc prèpei na epib˘loume th sunj kh $det(R) = +1$, pou orÐzei mia upoom˘da, thn $SO(3)$.

E.3 Om˘da Lorentz

H om˘da Lorentz sumbolÐzetai me $O(3,1)$. Ta stoiqeÐa thc om˘dac eÐnai oi 4×4 pÐnakec $\Lambda^\mu{}_\nu$ (touc opoÐouc ja anafèroume wc L) pou metasqhmatÐzoun ta 4-dianÔsmata: $x' = \Lambda x$. Gia na mènei analloÐwto to di˘sthma $x^\mu x_\mu$ prèpei na ikanopoieÐtai h sqèsh

$$g = \Lambda^T g \Lambda \tag{E.3.1}$$

ìpou g eÐnai h epÐpedh metrik me èna prìshmo meÐon ($g = -\eta$). Oi orj ìqronoi metasqhmatismoÐ Lorentz me orÐzousa +1 apoteloÔn epÐshc om˘da kai sumbolÐzontai me $SO(3,1)$. Oi orj ìqronoi metasqhmatismoÐ ikanopoioÔn thn sqèsh ${\Lambda^0}_0 \geq 0$. H om˘da Lorentz èqei èxi anex˘rthtec paramètrouc: treic gia tic peristrofèc kai treic gia tic wj seic.

Gr˘fontac ton metasqhmatismì Lorentz se apeirost morf $\Lambda = 1+\epsilon$ ìpou ta stoiqeÐa tou ε eÐnai apeirost˘ mikr˘ mporoÔme na gr˘youme thn sqèsh

$$\delta x^\mu \equiv x'^\mu - x^\mu = \epsilon^\mu_\rho x^\rho = \frac{i}{2}\epsilon^{\rho\sigma}L_{\rho\sigma}x^\mu \tag{E.3.2}$$

ìpou $L_{\rho\sigma}$ o genn torac twn metasqhmatism¸n Lorentz, pou dÐnetai apì thn sqèsh

$$L_{\mu\nu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu \tag{E.3.3}$$

H ˘lgebra thc om˘dac $SO(3,1)$ prosdiorÐzetai apì thn sqèsh

$$[L_{\mu\nu}, L_{\rho\sigma}] = ig_{\nu\rho}L_{\mu\sigma} - ig_{\mu\rho}L_{\nu\sigma} - ig_{\nu\sigma}L_{\mu\rho} + ig_{\mu\sigma}L_{\nu\rho} \tag{E.3.4}$$

EpÐshc mporoÔme na gr˘youme ton metasqhmatismì Lorentz sthn morf

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}T^{\mu\nu}} \tag{E.3.5}$$

ìpou $T^{\mu\nu}$

$$(T^{\mu\nu})_{\alpha\beta} = i({\delta^\mu}_\alpha {\delta^\nu}_\beta - {\delta^\nu}_\alpha {\delta^\mu}_\beta) \tag{E.3.6}$$

Oi èxi par˘metroi thc om˘dac apartÐzoun èna antisummetrikì pÐnaka 4×4: $\omega_{\mu\nu}$. PaÐrnontac ton metajèth katal goume sthn sqèsh E.3.4.

Gia na broÔme pwc metasqhmatÐzontai oi spÐnorec k˘tw apì metasqhmatismoÔc Lorentz prèpei na broÔme touc genn torec kat˘llhlhc diastatikìthtac pou na ikanopoioÔn thn sqèsh E.3.4. Gia touc 4-di˘statouc spÐnorec Dirac, h anapar˘stash twn gennhtìrwn epitugq˘netai me touc pÐnakec $S^{\mu\nu}$, oi opoÐoi orÐzontai wc ex c [31]

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{E.3.7}$$

Oi spÐnorec metasqhmatÐzontai me ton akìloujo trìpo

$$\Psi' = \Lambda_{\frac{1}{2}}\Psi = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}} \tag{E.3.8}$$

Gr˘fontac ton metasqhmatismì $\Lambda_{\frac{1}{2}}$ se apeirost morf mporoÔme na deÐxoume ìti

$$\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} = {\Lambda^\mu}_\nu\gamma^\nu \tag{E.3.9}$$

Aut h sqèsh eÐnai qr simh gia na apodeÐxoume ìti h lagkrazian Dirac eÐnai ametˆblhth kˆtw apì metasqhmatismoÔc Lorentz.

'Olec oi lagkrazianèc pou perigrˆfoun mia jemeli¸dh jewrÐa prèpei na eÐnai ametˆblhtec kˆtw apì metasqhmatismoÔc Lorentz [1].

[1] Οι λαγκραζιανές πρέπει επίσης να είναι αμετάβλητες κάτω από μετασχηματισμούς μεταφοράς που μαζί με τους μετασχηματισμούς Lorentz αποτελούν την ομάδα Poincare [31].

Par^rthma F

Paramètroi Feynman

H mèjodoc pou ja perigr^youme se autì to kef^laio metatrèpei diaforetikoÔc par^-gontec pou brÐskontai ston paronomast miac posìthtac se èna par^gonta uywmèno se k^poia dÔnamh. Autì eÐnai polÔ qr simo stouc upologismoÔc migadik¸n oloklhrwm^twn pou sunant^me stouc upologismoÔc diagramm^twn Feynman me brìqouc.

Arqik^ ja doÔme thn apl perÐptwsh me 2 par^gontec

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\big(xA+(1-x)B\big)^2} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{\big(xA+yB\big)^2} \tag{F.1}$$

'Opwc blèpoume arqik^ eÐqame dÔo par^gontec ston paronomast , A kai B, kai katal xame me èna par^gonta uywmèno sto tetr^gwno.

'Ena par^deigma efarmog c eÐnai

$$\frac{1}{(k-p)^2(k^2-m^2)} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{(k^2-2xk.p+xp^2-ym^2)^2} \tag{F.2}$$

OrÐzontac $\ell \equiv k - xp$ tìte o paronomast c exart^tai mìno apì to ℓ^2. To olokl rwma wc proc k metatrèpetai se èna olokl rwma wc proc ℓ, to opoÐo eÐnai polÔ pio eÔkolo, kaj¸c to ìrisma eÐnai sfairik^ summetrikì wc proc ℓ. Oi par^metroi x kai y onom^zontai par^metroi Feynman.

ParagwgÐzontac wc proc B brÐskoume ìti

$$\frac{1}{AB^n} = \int_0^1 dx \frac{ny^{n-1}}{\big(xA+(1-x)B\big)^{n+1}} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{\big(xA+yB\big)^2} \tag{F.3}$$

Qrhsimopoi¸ntac thn mèjodo thc epagwg c brÐskoume ìti

$$\frac{1}{A_1A_2\dots A_n} = \int_0^1 dx_1dx_2\dots dx_n\delta\left(\sum_{i=1}^n x_i - 1\right)\frac{(n-1)!}{(x_1A_1+x_2A_2\dots x_nA_n)^n} \tag{F.4}$$

Paragwgðzontac thn pio pˆnw èkfrash paðrnoume

$$\frac{1}{A_1^{m_1}A_2^{m_2}\dots A_n^{m_n}} = \int_0^1 dx_1 dx_2 \dots dx_n \delta\left(\sum_{i=1}^{n} x_i - 1\right)\frac{\prod_i^n x_i^{m_i-1}}{(\sum_i x_i A_i)^{\sum_i m_i}}\frac{\Gamma(\sum_i m_i)}{\Gamma(m_1)\dots\Gamma(m_n)} \tag{F.5}$$

Autìc o tÔpoc isqÔei kai sthn perðptwsh ìpou ta m_i den eðnai akèraioi [11].

Par^rthma G

Suntelestèc ìrwn stic entropĐec Renyi

To Đqnoc $\mathrm{Tr}(\rho_H)^m$, gia $m > 4$, perilamb^nei ton ìro $\mathrm{Tr}\,\rho_0$=1. Perilamb^nei epĐshc èna ìro $m\,\mathrm{Tr}\,\rho_0\epsilon$ afoÔ ìloi oi grammikoĐ ìroi wc proc ϵ èqoun to Đdio Đqnoc, anex^rthta apì thn seir^ tou pĐnaka ϵ wc proc ton pĐnaka ρ_0. EpĐshc perièqei èna ìro $m\,\mathrm{Tr}\,\rho_0\epsilon^2$ epeid up^rqoun m diaforetikoĐ ìroi me dÔo ϵ dĐpla-dĐpla.

Se t^xh ϵ^2 brĐskoume $\frac{m(m-3)}{2}\,\mathrm{Tr}\,\rho_0\epsilon\rho_0\epsilon$. Gia na broÔme ton suntelest , lÔnoume to akìloujo prìblhma sunduastik c: 'Estw ìti èqoume m jèseic kai jèloume na topojet - soume dÔo antikeĐmena (ϵ) se mh diplanèc jèseic. Up^rqoun m epilogèc na topojet soume to pr¸to kai $m-3$ epilogèc na topojet soume to deÔtero. DiairoÔme me to dÔo epeid den èqei shmasĐa poio ϵ topojetoÔme pr¸to.

Gia touc kubikoÔc ìrouc (ϵ^3) paĐrnoume touc ìrouc $m\,\mathrm{Tr}\,\rho_0\epsilon^3$ kai $m(m-4)\,\mathrm{Tr}\,\rho_0\epsilon^2\rho_0\epsilon$. Tèloc èqoume ton ìro $\frac{m(m-4)(m-5)}{6}\,\mathrm{Tr}(\rho_0\epsilon)^3$. Gia na broÔme ton suntelest autoÔ tou ìrou, prèpei na afairèsoume apì to sunolikì pl joc ìrwn t^xhc ϵ^3 pou isoÔtai me $\frac{m(m-1)(m-2)}{6}$ touc suntelestèc m kai $m(m-4)$. O teleutaĐoc ìroc mhdenĐzetai gia $m=5$ afoÔ gia na èqoume tètoio ìro qreiazìmaste na sundu^soume 6 pĐnakec ρ_0 kai ϵ.

H perĐptwsh twn ìrwn t^xhc ϵ^4 eĐnai pio perĐplokh. Arqik^ èqoume ton ìro $m\,\mathrm{Tr}\,\rho_0\epsilon^4$, kai touc ìrouc $m(m-5)\,\mathrm{Tr}\,\rho_0\epsilon^3\rho_0\epsilon$, $\frac{m(m-5)}{2}\,\mathrm{Tr}\,\rho_0\epsilon^2\rho_0\epsilon^2$. Oi teleutaĐoi dÔo ìroi eĐnai oi $\frac{(m-1)(m-5)(m-6)}{2}\,\mathrm{Tr}\,\rho_0\epsilon^2(\rho_0\epsilon)^2$ kai $P(m)(\mathrm{Tr}\,\rho_0\epsilon)^4$, ìpou $P(m)$ eĐnai o sunolikìc arijmìc ìrwn t^xhc ϵ^4, $\frac{m(m-1)(m-2)(m-3)}{4!}$ meĐon touc suntelestèc twn upìloipwn ìrwn

$$P(m) = \frac{m(m-1)(m-2)(m-3)}{4!} - \left(m + \frac{3m(m-5)}{2} + \frac{(m-1)(m-5)(m-6)}{2}\right) \tag{G.1}$$

Gia na broÔme ton suntelest tou $\mathrm{Tr}\,\rho_0\epsilon^2(\rho_0\epsilon)^2$ upojèsame ìti èqei thn morf poluwnÔmou tˆxhc m [1]. Sto ìrio ìpou $m \to \infty$, o suntelest c gÐnetai $\frac{m^3}{2}$. Epomènwc o suntelest c èqei thn morf $N(m) = \frac{(m-a)(m-b)(m-c)}{2}$. Gia na prosdiorÐsoume ta a, b, c qreiazìmaste treic timèc gia diaforetikˆ m. Oi pio aplèc peript¸seic eÐnai $N(6) = 0$, $N(7) = 6$, $N(8) = 21$. Efarmìzontac autèc tic timèc ston genikì tÔpo brÐskoume $(a, b, c) = (1, 5, 6)$.

[1] Απαραίτητο αφού ο συντελεστής πρέπει να είναι ακέραιος για ακέραια m

Bibliografía

[1] Theodore N. Tomaras and Nicolaos Toumbas. “IR dynamics and entanglement entropy”. In: *Phys. Rev. D* 101.6 (2020), p. 065006. doi: 10.1103/PhysRevD.101.065006. arXiv: 1910.07847 [hep-th].

[2] Susskind L. and Lindesay J. *An introduction to Black Holes, Information and the String Theory Revolution: The Holographic Universe*. World Scientific Publishing Company, 2005.

[3] EutÔqioc Kaïmakkˆmhc. *Κυματοσυνάρτηση Hartle-Hawking στην R2 θεωρία*. url: http://ucy.ac.cy/phy/documents/Documents/theses/undergraduate_theses/Eftychios_Kaimakkamis_dissertation_-_Final_draft.pdf.

[4] A. Strominger. “Lectures on the Infrared Structure of Gravity and Gauge Theory”. In: (2016). arXiv:1703.05448[hep-th].

[5] Steven Weinberg. *Gravitation and cosmology: principles and applications of the general theory of relativity*. first Edition. Cambridge University Press, 1972.

[6] S.W. Hawking. “Particle Creation by Black Holes”. In: *Commun. Math. Phys.* 43 (1975). Ed. by G.W. Gibbons and S.W. Hawking. [Erratum: Commun.Math.Phys. 46, 206 (1976)], pp. 199–220. doi: 10.1007/BF02345020.

[7] S.W. Hawking. “Black Holes and Thermodynamics”. In: *Phys. Rev. D* 13 (1976), pp. 191–197. doi: 10.1103/PhysRevD.13.191.

[8] Don N. Page. “Information in black hole radiation”. In: *Phys. Rev. Lett.* 71 (1993), pp. 3743–3746. doi: 10.1103/PhysRevLett.71.3743. arXiv: hep-th/9306083.

[9] Don N. Page. “Average entropy of a subsystem”. In: *Phys. Rev. Lett.* 71 (1993), pp. 1291–1294. doi: 10.1103/PhysRevLett.71.1291. arXiv: gr-qc/9305007.

[10] url: https://www.researchgate.net/figure/Schematic-figure-of-Page-curve-In-the-conjecture-entanglement-entropy-between-black_fig14_277145166.

[11] Daniel V. Schroeder Michael E. Peskin. *An Introduction To Quantum Field Theory*. Taylor & Francis Inc, 1995.

[12] Steven Weinberg. “Infrared photons and gravitons”. In: *Phys. Rev.* 140 (1965), B516–B524. doi: 10.1103/PhysRev.140.B516.

[13] F. Bloch and A. Nordsieck. "Note on the Radiation Field of the electron". In: *Phys. Rev.* 52 (1937), pp. 54–59. doi: 10.1103/PhysRev.52.54.

[14] F.E. Low. "Scattering of light of very low frequency by systems of spin 1/2". In: *Phys. Rev.* 96 (1954), pp. 1428–1432. doi: 10.1103/PhysRev.96.1428.

[15] T.H. Burnett and Norman M. Kroll. "Extension of the low soft photon theorem". In: *Phys. Rev. Lett.* 20 (1968), p. 86. doi: 10.1103/PhysRevLett.20.86.

[16] P.P. Kulish and L.D. Faddeev. "Asymptotic conditions and infrared divergences in quantum electrodynamics". In: *Theor. Math. Phys.* 4 (1970), p. 745. doi: 10.1007/BF01066485.

[17] Juan Martin Maldacena. "The Large N limit of superconformal field theories and supergravity". In: *Int. J. Theor. Phys.* 38 (1999), pp. 1113–1133. doi: 10.1023/A:1026654312961. arXiv: hep-th/9711200.

[18] Leonard Susskind. "The World as a hologram". In: *J. Math. Phys.* 36 (1995), pp. 6377–6396. doi: 10.1063/1.531249. arXiv: hep-th/9409089.

[19] Leonard Susskind, Larus Thorlacius, and John Uglum. "The Stretched horizon and black hole complementarity". In: *Phys. Rev. D* 48 (1993), pp. 3743–3761. doi: 10.1103/PhysRevD.48.3743. arXiv: hep-th/9306069.

[20] Andrew Strominger. "Black Hole Information Revisited". In: 2020, pp. 109–117. doi: 10.1142/9789811203961_0010. arXiv: 1706.07143 [hep-th].

[21] Daniel Kapec et al. "Infrared Divergences in QED, Revisited". In: *Phys. Rev. D* 96.8 (2017), p. 085002. doi: 10.1103/PhysRevD.96.085002. arXiv: 1705.04311 [hep-th].

[22] Daniel Carney, Laurent Chaurette, and Gordon Semeno. "Scattering with partial information". In: (June 2016). arXiv: 1606.03103 [hep-th].

[23] Daniel Carney et al. "Infrared quantum information". In: *Phys. Rev. Lett.* 119.18 (2017), p. 180502. doi: 10.1103/PhysRevLett.119.180502. arXiv: 1706.03782 [hep-th].

[24] V. Chung. "Infrared Divergence in Quantum Electrodynamics". In: *Phys. Rev. 140* (1965). doi: 10.1103/PhysRev.140.B1110.

[25] Barak Gabai and Amit Sever. "Large gauge symmetries and asymptotic states in QED". In: *JHEP* 12 (2016), p. 095. doi: 10.1007/JHEP12(2016)095. arXiv: 1607.08599 [hep-th].

[26] Sangmin Choi and Ratindranath Akhoury. "Subleading soft dressings of asymptotic states in QED and perturbative quantum gravity". In: *JHEP* 09 (2019), p. 031. doi: 10.1007/JHEP09(2019)031. arXiv: 1907.05438 [hep-th].

[27] Daniel Carney et al. "On the need for soft dressing". In: *JHEP* 09 (2018), p. 121. doi: 10.1007/JHEP09(2018)121. arXiv: 1803.02370 [hep-th].

[28] K¸stac Kìkkotac. *Γενική Θεωρία της Σχετικότητας*. url: https://www.astro.auth.gr/~kokkotas/lesson/book_gr.pdf.

[29] Joseph Polchinski. *String Theory, Vol. 1 (Cambridge Monographs on Mathematical Physics)*. Kindle Edition, 1998. Chap. Appendice.

[30] Sìlwn Zarkanðthc. *Θεώρημα Cauchy - Goursat*. url: http://www.catalysis.gr/mathematics/complex%5C%20functions/cauchy%5C%20-%5C%20goursat.html.

[31] Pierre Ramond. *Field Theory : A Modern Primer*. 2nd Edition. Westview Press; 2 edition, 2001.