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Abstract

We study electoral competition under the, so-called, Apostolic voting rule (AVR) in the framework of the Hotelling-Downs model (Osborne, 1993). The AVR is a two-stage election procedure composed of a voting stage and a lottery stage: the voters vote for the candidate they like best, and each of the two most-voted candidates is elected with even probability. Under standard assumptions regarding the voters' preferences, we show that the AVR leads to a unique –up to permutations of the players' identities– equilibrium: only two candidates enter in the electoral race and they choose distinct policy platforms. This is the first rule which is proved to support an essentially unique equilibrium in this popular model. Our analysis highlights that as long as candidates do not compete for a single first place (as in standard plurality or runoff elections), but for a number of them (as under the AVR), strategic incentives alter dramatically and lead to stable and predictable configurations.

Keywords: Apostolic Voting; Hotelling-Downs model; manipulation; lotteries; unique equilibrium.

JEL classification: D72

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"(23) And they appointed two, Joseph ... and Matthias. (24) And they prayed, and said, Thou, Lord, which knowest the hearts of all men, show whether of these two Thou hast chosen, (25) That he may take part of this ministry and apostleship, from which Judas by transgression fell, that he might go to his own place. (26) And they gave forth their lots; and the lot fell upon Matthias; and he was numbered with the eleven apostles." (Acts of the Apostles 1:23-26)

1 Introduction

According to the New Testament following Judas' departure the remaining eleven Apostles gave to Matthias the vacant position of the twelfth Apostle. The selection procedure that they used was two-staged: first the Apostles decided which two, out of about one hundred disciples, were the most suitable ones to fill the position –they decided that the most worthy ones were Joseph and Matthias– and then they drew lots and Matthias won.[1] This two-stage first-voting-then-lottery procedure took the name *Apostolic Voting* and was thereafter used for Patriarch elections by the churches of Russia and Serbia.

In this paper we analyze Apostolic Voting in the framework of the standard model of the economic theory of democracy: the Hotelling-Downs model of electoral competition. This setup is an extremely useful tool for the analysis of the performance of any electoral rule as it combines two important strategic dimensions of real world electoral competition: the dynamics of policy-platform competition when candidates care about winning and the decision of whether to enter the electoral race or not. In this setup it has been shown that, generically, i) there is no equilibrium[2] when elections are held under the plurality rule (Osborne, 1993) and that ii) many equilibria exist when elections are held under alternative multiple-vote or runoff rules (Haan and Volkerink, 2001; Brusco et al., 2012; Xefteris, 2016).[3] Hence, under all these popular voting rules the model cannot generate a clear prediction of the candidates' behavior; in the plurality rule case because of inexistence of equilibria and in the other cases due to multiplicity of equilibria.

We study the Hotelling-Downs model with an apostolic voting rule (AVR) and an arbitrary number of potential candidates and, under standard assumptions regarding the distribution of voters' preferences (mainly, log-concavity and uni-modality), we prove existence of an essentially unique equilibrium: only two candidates enter the electoral race and they choose distinct policy platforms. In this essentially unique equilibrium one entrant locates to the left and the other entrant locates –not necessarily equidistantly– to the right of the ideal policy of the median voter. That is, the two entrants' vote shares

[1]Ancient Israelites were very often drawing lots to reach a decision. See, for example, Lindblom (1962) for an account and analysis of mentions of lot casting in the Old Testament.

[2]The word "equilibrium" in this paper refers to a pure-strategy Nash equilibrium.

[3]One is referred to Bol et al. (2016) for a thorough literature review of endogenous candidacy models under alternative electoral rules.

need not be the same. To our knowledge this is the first rule which leads to stability in electoral competition, that is, to a unique equilibrium outcome in the framework of the standard model of the economic theory of democracy.

The AVR that we consider in this paper is identical to plurality rule as far as voters' participation is concerned: each voter casts one vote for the candidate who offers the policy platform that this voter likes most. The winner of the elections, though, is not necessarily the candidate who receives the largest vote share. Under an AVR the winner is determined by lot –that is, by an equiprobable draw– between the two candidates who received the largest vote shares. Hence, the voting stage determines a set of two candidates such that each gets elected with probability one half.

The main novel insight that our analysis provides can be summarized as follows: when candidates do not compete for a single top position, as they do under plurality, runoff or multi-vote procedures, but for a number of them, as under the described AVR, competition can lead to stable and predictable outcomes. In fact, a combined reading of all existing studies on the topic along with the current findings suggests that the prospects of equilibrium are enhanced both when voters have multiple ballots (either simultaneously, as in multiple-vote procedures, or sequentially, as in runoff elections) – and, hence, candidates do not care to be the top-ranked candidate in the preferences of any voter– and when candidates compete for many similar prizes –and, hence, candidates do not care to be the most-voted ones. Indeed, as we argue, the identified stabilizing effect of having essentially multiple winners does not restrict only to our specific AVR, but generalizes to broader classes of rules, some of which are pertinent in settings of applied interest. For instance, the *threshold rules*, that assign a fixed prize to each candidate that surpasses an exogenously set vote share, are used in interim stages of the democratic primaries to select which candidates get to participate in televised debates.[4] We argue that the equilibrium set of each of these rules contains the unique equilibrium of a given AVR. To support this claim we first extend our analysis to a general class of AVRs, by relaxing the assumption on the number of candidates that advance to the lottery stage, but, at the same time, imposing stricter assumptions regarding voters' preferences.

An AVR, apart from generically resulting in a unique equilibrium, is also found to have two additional attractive properties. First, it deals with *manipulation attempts* from external actors more efficiently than any conventional voting rule. By that we mean that under an AVR an external actor is less likely to determine the identity of the winner

[4]Moreover, it is not rare that hiring committees, or other collective bodies, have the mandate to propose a specified number of candidates to a higher authority which conducts the final selection. For example, the leader of the U.K. Conservative Party is elected in the following way: first, the interested party members declare candidacy, then the members of parliament that belong to the conservative party vote reducing the candidates to two, and, finally, all the party members vote between the two finalists and the leader emerges. While the final part of such a procedure is not explicitly a lottery stage, it might be viewed as such by the potential candidates and the first stage voters, when the preferences of the higher authority are her private information; making the whole process resemble the AVR.

than under any standard voting procedure. We study these manipulation attempts by the means of an extension to our original model in which a fixed and non-degenerate fraction of voters is committed to vote for a certain candidate independently of whose candidate's platform these voters like best. Arguably, such an extension captures the situation in which an external actor threatens or bribes a fraction of the voters to vote for her preferred candidate. This variation of the original model is of independent interest mostly because it relates to the reason why this rule was adopted for patriarch's elections by the Russian and Serbian church in the first place.[5] These decisions find support in the results of this variant of our model. Any rule that is consistent with majority rule in two candidate elections (this includes plurality, runoff rules and, essentially, all scoring rules) admits an equilibrium in which only the *advantaged* candidate enters and, thus, wins with certainty while an AVR always admits two-candidate equilibria: a disadvantaged candidate wins with a probability equal to one half. The second attractive property of the AVR is that the equilibrium that we characterize in this Hotelling-Downs setup exists also in citizen-candidate variations of the model. In other words, the identified equilibrium outcomes remains relevant in alternative, and equally plausible, environments of electoral competition.

In the remainder, we present the model (Section 2) and we formally characterize the essentially unique equilibrium of the game (Section 3). In the end of that section we moreover offer a brief explanation why a model with citizen-candidates and the same AVR would result in an identical equilibrium. Then (Section 4), we extend the model i) by considering that one candidate has an "advantage" (a fixed amount of voters always vote for this candidate independently of the policy platforms that candidates propose) and we show that AVR performs better than any conventional voting rule (plurality, runoff rules and, essentially, any scoring rule) in such a setup, and ii) by allowing that more than two candidates qualify for the lottery stage. Finally (Section 5), we discuss some of our central assumptions and how they allow us provide a broader interpretation to our findings, and we conclude (Section 6) by indicating promising directions for future work.

2 The model

We have a set of potential candidates who compete under an Apostolic voting rule (AVR) for an office. Formally, the set of potential candidates is $N = \{1, 2, ..., n\}$, where $n \in \mathbb{N}$. We consider that each voter has symmetric Euclidean preferences over $\mathbb{R}$ that are single-peaked. The ideal policies of a unit mass of voters are distributed according to an atomless, strictly increasing, twice-differentiable distribution function F with full sup-

[5]In the Appendix, included in the online Supporting Information file, we provide a brief presentation of the events which led to the implementation of an AVR by these two churches.

port over $[0,1]$. Given these assumptions, F has a unique median, $m \in (0,1)$, and a well-defined continuous density function denoted by f. The potential candidates simultaneously choose their strategies in the first stage of the game from the set $[0,1] \cup \{Out\}$. We are interested only in pure strategies and we denote by $y_i \in [0,1] \cup \{Out\}$ the choice of candidate $i \in N$. We define $K = \{i \in N | y_i \neq Out\}$. That is, if $i \in K$ then i is a candidate (not only a potential candidate). Voters observe the choices of the potential candidates and vote sincerely for the candidate who offers the policy platform nearest to their ideal policy. In case a voter is indifferent between two or more candidates, the voter evenly splits his vote between/among them. We define by $v_i(y_1, y_2, ..., y_n)$ the vote share of potential candidate $i \in N$ for the strategy profile $(y_1, y_2, ..., y_n)$. Obviously, if $y_i = Out$ then $v_i(y_1, y_2, ..., y_n) = 0$ independently of what the other players do. Finally, define $A = \{i \in N | v_i(y_1, y_2, ..., y_n) \geq v_j(y_1, y_2, ..., y_n)$ for any $j \in N\}$ and $B = \{i \in N \backslash A | v_i(y_1, y_2, ..., y_n) > v_j(y_1, y_2, ..., y_n)$ for any $j \in N \backslash A - \{i\}\}$.

The AVR is such that i) if $\#A > 1$ then each $i \in A$ is elected with probability $\frac{1}{\#A}$, ii) if $\#A = 1$ and $\#B = 1$ then each $i \in A \cup B$ is elected with probability $\frac{1}{2}$ and iii) if $\#A = 1$ and $\#B \neq 1$ then candidate $i \in A$ is elected with probability 1. Candidates are purely office-motivated; they maximize the probability of being elected. They moreover prefer the pure strategy *Out* to any pure strategy which gives them zero election probability and any strategy which gives them positive election probability to the pure strategy *Out*.[6]

When considering a particular strategy profile $(y_1, y_2, ..., y_n)$ we denote by $k = \#K$ (the number of actual candidates), by $0 \leq y^1 < y^2 < ... < y^r \leq 1$ the r distinct policy platforms that belong to the strategy profile $(y_1, y_2, ..., y_n)$ and by $n(y^j)$ the number of candidates who choose the policy platform y^j. As Osborne (1993), we consider that the left constituency of y^j is given by $F(y^j) - F(\frac{y^j + y^{j-1}}{2})$ for $2 \leq j \leq r$ and by $F(y^j)$ for $j = 1$ and that the right constituency of y^j is defined symmetrically. The constituency of y^j is the sum of its left and its right constituency.

3 Equilibrium analysis

We construct our main result by proving a series of lemmata.[7] We start by arguing that every equilibrium should involve more than one candidate.

Lemma 1. *If $n \geq 2$ then for any F there exists no equilibrium with $k \leq 1$.*

Indeed, if at most one player declares candidacy then a non-entrant prefers to enter the race by choosing any possible policy. In this way she receives a positive vote share

[6] Apart from the employed AVR, our model is compatible with the one proposed by Osborne (1993). This ensures direct comparability of our findings to the findings of other studies that used a similar framework to analyze different electoral rules, like Haan and Volkerink (2001), Brusco et al. (2012), and Xefteris (2016).

[7] The proofs of all the results of this section are in the Appendix.

and she is elected with a probability at least equal to one half.

The next two lemmata relate, mainly, to the platform selection dimension of the game, and less to its endogenous candidacy aspect. Hence, they are supported by arguments that are similar to those employed by exogenous candidacy models, such as Cox (1987).

Lemma 2. *If $n \geq 3$, then in any equilibrium with $k \geq 3$ it is the case that: a) $n(y^j) \leq 2$ for all j, b) $n(y^1) = n(y^r) = 2$, c) if $n(y^j) = 2$ then the left and the right constituency of y^j are equal and d) the vote share of all $k \geq 3$ candidates who enter is $\frac{1}{k}$.*

Configurations with three or more entrants need to satisfy all these conditions to constitute an equilibrium. Evidently, all players that declare candidacy must be elected with positive probability (or else, they would prefer to stay out of the race), and for this to happen, it is necessary that they all tie in the first place. Moreover, extreme entrants must announce the same platform with at least one other entrant, otherwise they would have incentives to approach the other entrants and increase their election probability. Interestingly, these conditions are too stringent and, as the next lemma establishes, they cannot be satisfied in generic cases.

Lemma 3. *For no F is there a strategy profile which satisfies the four conditions of Lemma 2 for $k = 3$. For almost no F is there a strategy profile which satisfies the four conditions of Lemma 2 for $k \geq 4$.*

This suggests that if an equilibrium exists, it has to involve, generically, only two entrants. Given that our n-player game is symmetric, it directly follows that if a two-candidate equilibrium exists then many similar two-candidate equilibria exist that differ only in the identities of the active candidates and not in the strategies that the active candidates employ. If we assume that in a two-candidate equilibrium we have that $0 \leq y_1 \leq y_2 \leq 1$ and that $y_i = Out$ for any $i \geq 3$ then we can significantly reduce the complexity of exposition of the results that follow without any loss of generality.

We start with an immediate result, which simply states that if only two players enter, they are indifferent among all possible policy platforms.

Lemma 4. *Any strategy profile $(y_1, y_2, Out, Out, ..., Out)$ with $(y_1, y_2) \in [0,1]^2$ is such that the two active candidates get the same payoff ($\frac{1}{2}$).*

To simplify exposition of our main results we define the following set of strategy pairs:

$$\mathcal{Y} = \{(y_1, y_2) \in [0,1]^2 \text{ s.t. (A) } 2F(y_1) = F(\tfrac{y_1+y_2}{2}), \text{ (B) } 2[1 - F(y_2)] = 1 - F(\tfrac{y_1+y_2}{2}) \text{ and (C) } F(\tfrac{y_1+y_2}{2}) \in (\tfrac{1}{3}, \tfrac{2}{3})\}.$$

This set of strategy pairs will prove particularly useful for equilibrium characterization, since, they are the only platform pairs that can be part of a two-candidate equilibrium.

Lemma 5. *If $n \geq 3$ and the strategy profile $(y_1, y_2, Out, Out, ..., Out)$ such that $y_1 \leq y_2$ is an equilibrium of the game, then $(y_1, y_2) \in \mathcal{Y}$.*

Hence, an equilibrium with two entrants must be such that when a third candidate enters i) she gets the lowest vote share (among the three active candidates) and ii) at least one of the other two candidates gets a vote share strictly larger than hers. It is noteworthy that in the literature which considers electoral competition between two established candidates and a potential entrant (initiated by Palfrey, 1984) one can find models which consider that a third candidate enters only if she can secure a vote share strictly larger than the vote share of at least one of the established candidates (see, for example, Greenberg and Shepsle, 1987; Rubinchik and Weber, 2007). It has actually been proven by Greenberg and Shepsle (1987) that the strategies of the two established candidates in such games must also satisfy the three conditions which characterize the elements of set $\mathcal{Y}$. This observation is extremely useful as it allows us skip the proof of the technical lemma that follows.

Lemma 6. *If F is unimodal and strictly log-concave then $\mathcal{Y}$ is a singleton, and its unique element is denoted by $(\dot{y}_1, \dot{y}_2)$.*

We must stress here that it is very easy to prove that a pair $(\dot{y}_1, \dot{y}_2) \in [0, 1]^2$ which satisfies $2F(\dot{y}_1) = F(\frac{\dot{y}_1+\dot{y}_2}{2})$ and $2[1 - F(\dot{y}_2)] = 1 - F(\frac{\dot{y}_1+\dot{y}_2}{2})$ exists for every absolutely continuous F. Such a pair obviously also satisfies $F(\frac{\dot{y}_1+\dot{y}_2}{2}) \in (\frac{1}{3}, \frac{2}{3})$ in all but very skewed Fs. Hence, the set $\mathcal{Y}$ is guaranteed to be non-empty quite generally. This means that the assumptions that F is unimodal and log-concave are more important in guaranteeing uniqueness of such a pair $(\dot{y}_1, \dot{y}_2) \in [0, 1]^2$ than establishing existence. In other words, since Lemma 5 dictates that for the strategy profile $(\dot{y}_1, \dot{y}_2, Out, Out, ..., Out)$ to be an equilibrium it has to be the case that $(\dot{y}_1, \dot{y}_2) \in \mathcal{Y}$, our observation implies that unimodality and log-concavity of F are more important in guaranteeing uniqueness of an equilibrium such that only two candidates enter the race than guaranteeing existence of such an equilibrium.

Our next result establishes that, as far as policy is concerned, all two-candidate equilibria are identical. That is, while the set is not a singleton, each two-candidate equilibrium induces the same distribution over policy outcomes.

Lemma 7. *For any unimodal and strictly log-concave F the game admits a unique –up to permutations of the players' identities– two-candidate equilibrium, which is given by the strategy profile $(\dot{y}_1, \dot{y}_2, Out,\ Out, ..., Out)$.*

Notice that despite the similarities with the entry deterrence literature, our result is much more general. In our case the equilibrium exists for any unimodal and strictly log-concave distribution of voters' ideal policies while the equilibrium of the Greenberg and Shepsle (1987) and the Rubinchik and Weber (2007) entry deterrence games moreover requires that F is such that $F^{-1}(\frac{1}{4}) + F^{-1}(\frac{3}{4}) = 2F^{-1}(\frac{1}{2})$. This is a very restrictive requirement. For example, our result holds for *any* unimodal Beta distribution (that is, for any Beta distribution with parameters $\alpha, \beta > 1$) while the equilibrium of the entry deterrence games exists for *almost no* Beta distribution (it exists, essentially, only for parameter values $a = \beta > 1$).

The reason why the equilibrium under AVR is generic and the equilibrium of the entry deterrence model with plurality rule is not, lies in the following fact: the unique pair of locations which satisfies the three conditions which characterize the elements of set $\mathcal{Y}$ is such that $\frac{\dot{y}_1+\dot{y}_2}{2} \neq m$ for any F which does not satisfy $F^{-1}(\frac{1}{4}) + F^{-1}(\frac{3}{4}) = 2F^{-1}(\frac{1}{2})$ (for example, for any Beta Distribution with a not perfectly symmetric density). In the entry deterrence literature such a pair cannot be part of an equilibrium since $\frac{\dot{y}_1+\dot{y}_2}{2} \neq m$ implies that the two established candidates get asymmetric vote shares and, hence, the established candidate who expects to be least voted has incentives to deviate. In elections, though, under an AVR this asymmetry between the expected vote shares of the two entrants is not a problem at all. Since they are the only two candidates that are expected to run for the office and since the pair of locations guarantees to each of them a non-degenerate vote share none of these two candidates may increase her expected payoff ($\frac{1}{2}$) by deviating to any other location.

We are ready to state our main result.

Proposition 1. *If $n \geq 3$ then for almost all unimodal and log-concave F's there exists a unique —up to permutations of the players' identities– equilibrium which is given by the strategy profile $(\dot{y}_1, \dot{y}_2, Out, Out, ..., Out)$.*

Consider, for example, that $F(x) = x^2$ for $x \in [0, 1]$ (the triangular distribution with a mode at 1). Then, we will have that i) the first candidate will locate about .46 and will get a vote share approximately equal to .42, ii) the second candidate will locate about .84 and will get a vote share approximately equal to .58, and iii) no third candidate will enter the race: each of the two entrants has an election probability equal to $\frac{1}{2}$.

It is important to note here that the identified equilibrium would still exist if we considered a citizen-candidate variation of the model with the same AVR.[8] To see why this is true consider the same set of assumptions as here with the following modifications: there is no distinct set of potential candidates and each of the infinitely many citizens (who just care about the implemented policy) can declare candidacy at a cost $c \geq 0$. The policy platform of this citizen-candidate will coincide with her ideal policy like in all citizen-candidate models and each citizen declares candidacy if and only if being a candidate increases the policy-dependent part of her expected utility by more than c. When c is sufficiently small, then an equilibrium of this game is such that exactly one

[8]It is known that citizen-candidate models are by construction prone to admitting multiple equilibria (see, for example, Dhillon and Lockwood, 2002) so we make no claims about uniqueness.

citizen with ideal policy $\dot{y}_1$ and exactly one citizen with ideal policy $\dot{y}_2$ declare candidacies and no other citizens become candidates. This is an equilibrium because i) if the citizen-candidate located at $\dot{y}_1$ ($\dot{y}_2$) exits then the policy outcome changes from the fair lottery between $\dot{y}_1$ and $\dot{y}_2$ to the sure outcome $\dot{y}_2$ ($\dot{y}_1$) which is strictly worse for her, ii) if a citizen with ideal policy $y \in [0, \dot{y}_1) \cup (\dot{y}_1, \dot{y}_2) \cup (\dot{y}_2, 1]$ decides to become a candidate then she will end up having a vote share strictly smaller than the vote share of each of the other two citizen-candidates and thus the policy outcome (the fair lottery between $\dot{y}_1$ and $\dot{y}_2$) will remain unchanged by her decision to run in the elections, and iii) if a third citizen with ideal policy $\dot{y}_1$ ($\dot{y}_2$) declares candidacy then she will tie in the last position with one of the first two citizen-candidates (with the one who also has ideal policy $\dot{y}_1$ ($\dot{y}_2$)) and, hence, the policy outcome will change from the fair lottery between $\dot{y}_1$ and $\dot{y}_2$ to the sure outcome $\dot{y}_2$ ($\dot{y}_1$) which is strictly worse for her.

This equilibrium survives, for obvious reasons, even if we consider i) the exact citizen-candidate preferences (a mixture of office- and policy- motivation and a sufficiently small, but positive, candidacy cost) of Osborne and Slivinski (1996) or ii) co-existence of some office-motivated potential candidates and of citizen-candidates (Dziubinski and Roy, 2013).[9]

4 Extensions

4.1 A model with an advantaged candidate

In this section we assume that there is a mass of voters with measure $\delta \in (0, 1)$ who always vote for the *advantaged* candidate, player 1, when the advantaged candidate is participating in the elections (when the advantaged candidate does not participate in the elections we assume that they behave like the rest of the voters).[10] We assume that the ideal policies of this bribed/threatened sub-electorate are distributed according to some distribution function $\hat{F}$ on $[0, 1]$. The remaining voters, which amount to a mass $1 - \delta$ and whose ideal policies are distributed according to a twice differentiable distribution function F on $[0, 1]$ behave exactly as in the main part of paper: they vote for the candidate who offers the policy which they like most.

We will argue that the AVR that we considered above (a lottery between the two most voted candidates) i) never results in an equilibrium in which only the advantaged candidate enters and ii) it results in equilibria in which at least one disadvantaged candidate enters. That is, the equilibrium election probability of a advantaged candidate cannot exceed $\frac{1}{2}$. More importantly, we will argue that standard majority-consistent rules have

[9] Details can be obtained from the authors upon request.

[10] The term *advantaged candidate* has been used in the literature to describe alternative non-policy advantages that certain candidates enjoy, like those related to their valence or quality features (Aragones and Palfrey, 2002; Hummel, 2010; Aragones and Xefteris, 2012, 2017).

equilibria in which only the advantaged candidate enters and wins with probability one. That is, an AVR is found to truly serve the reason for why it was adopted for patriarch's elections by the Russian and the Serbian church in the first place: it resists external attempts of outcome manipulation more efficiently than standard rules.

Definition 1. *We say that a decision rule is majority-consistent if and only if its outcome in a binary social choice problem coincides with the majority rule outcome.*

Notice that this is in fact a very broad category of decision rules. It includes runoff rules, plurality rule, Borda count and many others.

Proposition 2. *When elections take place according to a majority-consistent rule then there exists an equilibrium in which only the advantaged candidate enters (and thereafter wins with probability one) for any $\delta \in (0,1)$ and any F.*

The reasons which support this proposition are quite obvious. If the advantaged candidate is expected to enter and to locate at the position of the median voter (or sufficiently near) then a potential entrant knows that by entering she can at most secure the votes of $\frac{1-\delta}{2} < \frac{1}{2}$ voters and, hence, lose with certainty. That is, any majority-consistent rule supports such one-candidate equilibria in which the advantaged candidate wins with certainty. In other words, even if a tiny subset of the voters is bribed/threatened to support a advantaged candidate, these rules support equilibria which are such that this advantaged candidate wins with certainty.

Proposition 3. *When elections take place according to the AVR then: a) no one-candidate equilibria exist for any F and b) when F is unimodal and strictly log-concave then we have two-candidate equilibria such that the advantaged and a disadvantaged candidate run in the elections.*

The AVR by construction provides incentives for a disadvantaged candidate to enter the race when only the advantaged candidate is expected to enter for any $\delta \in (0,1)$. This is so because a potential entrant knows that by actually entering the race at any location she will get a positive vote share and, hence, an election probability equal to $\frac{1}{2}$. It is thereafter natural that no one-candidate equilibria exist. To see why two-candidate equilibria exist just notice that if F is unimodal and strictly log-concave and the advantaged candidate and a disadvantaged candidate locate at $(\dot{y}_1, \dot{y}_2) \in [0,1]^2$ (the pair of locations which characterize the elements of set $\mathcal{Y}$) then a potential entrant expects a vote share strictly smaller than the vote share of the advantaged candidate and a vote share weakly smaller than the vote share of the active disadvantaged candidate independently of where she will locate. That is, the AVR that we analyzed will assign her the office with probability zero and the existence of two-candidate equilibria is guaranteed.

To sum up, any majority-consistent rule has equilibria in which the advantaged candidate wins with probability one while any equilibrium of the AVR assigns the office to the advantaged candidate with a probability at most as large as $\frac{1}{2}$. This reasons with the adoption of AVRs by the Russian and the Serbian church, since –at the time of these decisions– they were experiencing severe pressures from the political regimes to elect leaders to the political regimes' liking.

Of course, the increased robustness of the AVR to manipulation attempts might come at a cost in terms of outcome efficiency. This is true especially when the influence of the external actor, δ, is small. In such cases the one-candidate equilibria of majority-consistent rules are such that the unique entrant proposes a policy close to the median voter, and, hence, she does not dissatisfy any of the voters too much, while under an AVR the policy outcome will most likely be more extreme. Notice though, that the opposite can also be true when the external actor is very influential, i.e. when δ is large. In such cases, majority-consistent rules support one-candidate equilibria such that the unique entrant proposes a very extreme policy; while the AVR can deliver more moderate outcomes with a probability equal to $\frac{1}{2}$. All these observations suggest that the relationship between manipulation-resistance and outcome extremity is anything but trivial, and should receive serious consideration before any electoral reform.

4.2 General AVRs

We now extend our analysis to apostolic voting rules with a lottery stage among $h \geq 3$ candidates. All the assumptions and the notation remain the same, except from some required modifications and simplifications.

To focus only on equilibria that do not depend on the size of the pool of potential entrants we assume that $N = \mathbb{N}^+$. For each potential candidate $i \in N$, and for each strategy profile $(y_1, y_2, ...)$, we define $H_i = \{j \in N | v_j(y_1, y_2, ...) \geq v_i(y_1, y_2, ...)\}$, i.e. the set of candidates which perform at least as well as i, including i. The $h-$AVR is such that i) if $\#H_i \leq h$ for some $i \in N$, then each $i \in N$ with $\#H_i \leq h$ is elected with probability $\frac{1}{\#\{i \in N | \#H_i \leq h\}}$, and ii) if $\#H_i > h$ for each $i \in N$, then each $i \in N$ with $\#H_i = \min\{\#H_1, \#H_2, ...\}$ is elected with probability $\frac{1}{\#\{j \in N | \#H_j = \#H_i\}}$.[11] Finally, we consider that the distribution of voters' ideal policies is uniform over the policy space. While this is obviously a restriction compared to the more general distributions considered for our main analysis, we note that our results are not cut-edge. That is, as we argue in the end of this section, they continue to hold as log as the voters are similarly concentrated in any two subsets of the policy space of similar length. Moreover, the uniform distribution is a case in which all popular rules lead to their generic results (i.e. no equilibrium exists under plurality rule, while multiple exist under runoff and multi-vote schemes).[12]

[11] This is with a slight abuse of notation. When H_i is infinite for every $i \in N$, then each player is elected with probability zero.

[12] One is referred to Osborne (1993), Brusco et al. (2012), and Xefteris (2016).

So, even if the need to generalize one aspect of our analysis (the number of candidates that qualify for the lottery stage), requires a simplification in another part (the distribution of the voters' ideal policies), the employed approach still allows meaningful comparisons with popular voting rules and relies on an assumption used in several seminar papers in electoral competition literature (e.g. Aragones and Palfrey, 2002; Hummel, 2010).

In a fashion similar to our main formal analysis, we characterize the equilibria of this extension of the model by proceeding in steps. In the Appendix, we fully develop the proofs of the statements that require novel arguments, and rely on previously proved results for the rest.

We start by noticing that, in equilibrium, the number of entrants must be at least as large as h.

Lemma 8. *There exists no equilibrium with $k \leq h-1$.*

Indeed, if less than h players declare candidacy, a non-entrant has incentives to deviate and announce any policy. In this way she gets a positive vote share, qualifies to the lottery stage and wins with positive probability.

Lemma 9. *In any equilibrium with $k \geq h+1$ it should be the case that a) $n(y^j) \leq 2$ for all j, b) $n(y^1) = n(y^r) = 2$, c) if $n(y^j) = 2$ then the left and the right constituency of y^j are equal and d) the vote share of each candidate who decides to enter is $\frac{1}{k}$.*

Like in the benchmark case, in any equilibrium with more than h entrants it is true that all entrants have identical vote shares, and they are located in a way that disincentivizes the entry of additional candidates. Interestingly, any configuration that satisfies the presented conditions cannot be an equilibrium: If only one candidate is located at y^2 then by conditions b) and c) of the above lemma, this candidate is voted by more voters than either of the candidates located at y^1, violating condition d). If exactly two candidates are located at y^2 then a non-entrant can deviate and enter between y^1 and y^2, get a vote share equal to $\frac{1}{k}$ and be elected with positive probability.

Lemma 10. *A strategy profile that satisfies the conditions of Lemma 9 is not an equilibrium.*

Hence, in equilibrium the number of entrants must be identical to h. The next lemma presents some conditions that need to hold in such an equilibrium.

Lemma 11. *In any equilibrium with $k = h$ it should be the case that: a) $n(y^j) = 1$ for all y^js, b) the left and the right constituency of y^j are equal for all j, and c) the vote share of each candidate who decides to enter is $\frac{1}{k}$.*

We now have all the required tools to characterize the equilibria of this general class of AVR voting rules.

Proposition 4. *The game admits a unique —up to permutations of the players' identities– equilibrium and it is such that* $(y_1, y_2, ..., y_h, y_{h+1}, ...) = (\frac{1}{2h}, \frac{3}{2h}, ..., \frac{2h-1}{2h}, Out, ...)$.

This result relies on the fact that, for any $h \geq 3$, when only h candidates enter, they can always locate in such a way that any non-entrant cannot gain a strictly larger vote share than any of the entrants, and, hence, cannot advance to the lottery stage. We need to stress that while our arguments make use of the fact that the distribution of the voters is uniform, such configurations are plausible in much more general cases. To see this notice that for any fixed h and any sufficiently uniform distribution –that is, a log-concave distribution which satisfies $f(x) < sf(y))$ for every $(x, y) \in [0, 1]^2$, for a sufficiently small $s > 1$– we can find a profile $(y_1, y_2, ..., y_h, Out, ...)$ with $y_1 < y_2 < ... < y_h$ such that for every active candidate the left constituency of her proposal is equal to its right constituency.[13] Additionally, in such cases all entrants receive similar vote shares, and, hence, none of the non-entrants has any incentives to mimic any entrant, or to locate in between any two of them.

5 Discussion

In this section we focus on a key characteristic of the analyzed AVRs: to qualify for the lottery stage a candidate needs to be clearly among the specified number of top positions, and not merely tie with a number of other candidates for the last admissible spot. This assumption requires that the candidates that get a chance to win are never more than a predetermined number –except, of course, when a larger number of candidates ties in the first place and there is no reasonable way to select only a subset. This requirement disincentivizes non-entrants from mimicking the decisions of the entrants when the number of entrants equals the number of candidates that advance to the lottery stage. Notice, though, that this is so only when voters perfectly split their votes between two candidates that offer the same policy. In a more general model in which different voters would be allowed to vote for different candidates with different probabilities when they are policy-wise indifferent between them, candidates that offer the same policies would not necessarily tie, and hence one could easily relax our qualification requirement.

Consider for simplicity the variation of the model in which each candidate is characterized by a number of non-policy features in the spirit of Krasa and Polborn (2012, 2014) that affect voters' behavior only when they are indifferent among the platforms proposed by a number of candidates as in Aragones and Palfrey (2002). That is, voters vote for the

[13]Indeed, for such distributions, it is easy to show that for every $y_2 \in (0, 1)$ one can find $y_1(y_2) < y_2$ such that $F(y_1(y_2)) = F(\frac{y_1(y_2)+y_2}{2}) - F(y_1(y_2))$, and $y_1(y_2)$ is continuously increasing in y_2 with a slope less than one. One can similarly show that for every $y_3 \in (0, 1)$ one can find $y_2(y_3) < y_3$ such that $F(y_2(y_3)) - F(\frac{y_2(y_3)+y_1(y_2(y_3))}{2}) = F(\frac{y_2(y_3)+y_3}{2}) - F(y_2(y_3))$, and $y_2(y_3)$ is continuously increasing in y_3 with a slope less than one, and so on and so forth up to candidate h.

candidate that proposes the platform they like best, and when they are policy-wise indifferent among a number of candidates they vote on the basis of the candidates' immutable characteristics like valence, experience, gender, or race. In such a case, even if one allowed a candidate to be elected with positive probability when she ties for the last qualification place with another candidate, equilibria similar to the ones that we identified could still exist and be unique. In fact, if candidates can be ordered by their minimal non-policy advantage (e.g. when $y_i = y_{i'}$ with $i < i'$, then both candidates get a positive vote share but candidate i gets more votes than i'), then there should exist a unique equilibrium in which only the first two players enter the race.

So while our qualification requirements are not inconsequential, it is important to see that they can be relaxed under plausible assumptions regarding voters' tie-breaking behavior.[14] Perhaps more importantly, these assumptions make the analysis of AVRs relevant to broader contexts of applied interest, including elections under, what we call, the *threshold rules*. Consider a situation in which a candidate needs to pass a certain vote share threshold in order to gain a fixed prize. Such electoral competition environments are, strategically, very similar to the AVRs: in both kinds of systems candidates do not care about their exact vote share but only whether it is above a bar or not. In the case of the AVRs this bar is a specific position in the order of candidates' vote shares, while in the case of threshold rules this bar is an exogenously set vote share. If for instance, a threshold rule requires from a candidate a vote share larger than $t \in (0, 1)$ to enjoy the prize, then the equilibrium of the $h-$AVR with $\frac{1}{h+1} < t < \frac{1}{h}$ is an equilibrium under this threshold rule, too.[15] Notice that if we had employed alternative qualification requirements for the AVRs and allowed ties for the last position to give a positive probability of election, this mapping between the AVRs and the thresshold rule would break, making the appeal of our analysis less general. Evidently, threshold rules penalize last-position ties: if for instance $t = .18$ and five candidates enter at $(\frac{1}{10}, \frac{3}{10}, \frac{5}{10}, \frac{7}{10}, \frac{9}{10})$, then a non-entrant by entering cannot secure more than 10% of the total votes, and hence she prefers to stay out.

While AVRs are not among the most popular voting systems, it is easy to argue that the threshold rules are relevant in several contexts of empirical interest. In the process for presidential nomination of the democratic party there are several stages at which, at least temporarily, the candidates fight to pass a survival/viability threshold. For instance, only candidates that pass a certain level of support –which can be, sometimes, as high as 7%– in an authorized poll are allowed to participate in a televised debate, and hence be

[14]Notice that when different potential candidates enjoy a different minimal non-policy advantage, an equilibrium may exist not only under the variation of the AVR in which the ties for the last qualification place lead to a positive probability of election, but, also, under standard plurality rule. While the analysis of such a model is out of the scope of this paper, it presents as an interesting avenue for future research.

[15]Indeed, if the players employ the same profile, all entrants pass the threshold and enjoy the prize, and all non-entrants cannot pass the threshold if they enter at any location.

viable contenders. Moreover, only candidates that get more than 15% of votes in a given state elect delegates for the party convention. In general election contexts, third party or independent candidates need to reach support thresholds validated by signatures to have their name on the ballot. Since, such candidates participate more often to gain media attention (for personal or other goals) than to beat the mainstream party candidates, it is plausible that they compete in the signature collection stage with other independent candidates for the support of a pool of active voters (i.e. voters that would be willing to sponsor an independent candidate if their platform is close to their ideal policy), and those that manage surpass a threshold of required support –which, in certain states, is beyond 14.000 signatures (McDonald and Samples, 2007)– gain the attention boost that they are after. Of course, in many of these cases the incentives to surpass the threshold are not the only ones present –affecting the implemented policy, or gaining more support compared to the other entrants might still be valuable– but it could be argued that, at least in some of them, it is the predominant goal, and hence the one that is mostly responsible for candidates' choices.

6 Concluding remarks

We have analyzed a model of electoral competition with endogenous entry and platform selection under an AVR and we have proved that essentially a unique two-candidate equilibrium exists under fairly standard assumptions regarding voters' preferences. This is a positive feature of AVR compared to alternative more conventional rules which fail to generate a unique equilibrium prediction. Observe, though, that the appeal of an electoral rule depends on how a society values its equilibrium properties, and not on these properties per se. So is the existence of a unique equilibrium desirable with respect to conventional welfare criteria?

While its relationship with classic normative goals, like utilitarian welfare or fairness, is not clear, the existence and uniqueness of an equilibrium seems to comply with a possible desire of the designer to minimize strategic (or endogenous) uncertainty (Knight, 1921). Such uncertainty is present in games that do not admit a unique equilibrium outcome and has been shown to be disliked by real subjects in experimental settings (see, for example, Heinemann et al., 2009). A voting rule like the AVR alleviates the designer from such concerns as it leads to a unique distribution over policy outcomes. To be sure, voting rules that admit no equilibrium, or several of them, might perform better than the AVR in other welfare-related dimensions. It would be interesting to investigate experimentally the performance of the AVR and other rules in the laboratory, not only to test the equilibrium predictions of the theoretical models, but also to compare their welfare properties and their appeal to actual voters.

Another aspect of AVR that seems to be quite different to any other applied rule, is the strategic-voting incentives that it provides to the concerned electorate. Indeed, the standard Hotelling-Downs model works with sincere voters (Osborne, 1993), but strategic-voting models with a fixed set of alternatives are known to generate different behavioral predictions under alternative voting rules. For instance, plurality rule usually leads to strategic coordination around two candidates –to the so-called Duvergerian equilibria (Palfrey, 1988; Xefteris, 2019)– while this need not be so when voters vote according to runoff rules or approval voting (Bouton, 2013; Nunez, 2010). How would an AVR shape the behavior of strategic voters in environments with several alternatives? Would it lead to two-candidate equilibria like plurality voting, or would it generate outcomes with several alternatives receiving non-degenerate vote shares. Evidently the treatment of such a question is beyond the scope of the present analysis, but it appears as a promising direction for future study.

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7 Appendix

7.1 Apostolic voting in the Russian and the Serbian church

Peter the Great, the ruler of Russia, abolished the position of the leader of the church in 1700, appointing the Most Holy Governing Synod as the main governing body of the church. This committee, composed of bishops and tsar's bureaucrats, helped the Russian rulers control the church for two centuries. During the murky days preceding the Russian revolution, and in particular with the abdication of Nicholas II in 1917, an opportunity presented itself to attenuate the rulers' influence in church matters. The Russian church officials decided it was time that they alone elect the head of the church, the patriarch, in order to fortify the Church's position in these times of political turmoil. "The Church is drawing into a war", said the Archbishop of Kishnyev, Anastasiy on October 11 th 1917, "Therefore, it needs a leader."[16] However, a somewhat unusual method of election was introduced. Though several church officials stood in protest to it, it was decided that the head of the church would be elected by a two-step procedure, called *Apostolic Voting*. In the first step, three candidates are elected among all the eligible bishops, and in the second, the winner is drawn by lottery, just the way the twelfth apostle, Matthias, was chosen to fill Judas Iscariot's vacated position. Apparently, this method was selected in order to make sure that the election of the church leader would remain devoid of any outside meddling, leaving it up to "the Holy Spirit" to decide who will henceforth lead the Russian church.

Forty years later, in 1958, the Serbian Orthodox Church was to elect its 43rd patriarch. Several days prior to the elections, the president of the religious committee and member of Tito's government gathered all bishops in the Patriarchate and informed them straightforwardly that the Government would like to see bishop German as the new patriarch. He also suggested that there should not be any arguments against it, stating that he "came here to settle this matter as friends."[17] German won the elections and was thereafter called the *red patriarch*. In 1967 the Serbian church adopted an AVR for patriarch's election to surmount such attempts of secular authorities to determine who is to be the head of the church. The precise AVR that they use is a bit more complex than the one we have in mind but it is still based on the same first-voting-then-lottery principle. The election procedure is the following: all members of the Episcopal Council vote for three candidates, from the list of eligible ones, in as many rounds as it takes to elect the candidate with an absolute majority of votes. Once they have the first name, it is stricken from the list and they proceed with voting for the second candidate in the same manner. Finally, the names of the three elected candidates are publicly noted, put into three different envelopes, sealed, shuffled and placed into a chest. Then after the

[16]The Acts of the Holy Council of Russian Orthodox Church 1917-1918.

[17]For a detailed presentation of all these events one is referred to Radic (2002, 2009).

Holy Liturgy, a selected monk takes them out, picks one of the three envelops and reads aloud the name of the new patriarch. It is noteworthy that the Serbian church employed the described AVR even in the most recent patriarch election.

7.2 Proofs

Proofs of the results presented in Section 3

Proof of Lemma 1. This follows directly from the definition of the AVR. Assume first that an equilibrium with $k = 0$ exists. Then every player is elected with even probability, but if player i enters, then i is elected with certainty. Hence, every player wants to deviate and such a profile cannot be an equilibrium. Now consider that an equilibrium profile exists such that only the first player enters the electoral race. Then, the probability of election of the second player is, naturally, zero. If the second player deviates and enters the race by proposing the same platform as the first player, then she will receive a strictly positive vote share. Therefore, she will get the office with probability $\frac{1}{2}$; the deviation is profitable and hence, such an equilibrium cannot exist. □

Proof of Lemma 2. We start by observing that in equilibrium all entrants enjoy a strictly positive election probability. By $k \geq 3$ and the definition of the AVR, we have that all entrants enjoy a strictly positive probability of election only if all of them receive the same vote share $\frac{1}{k}$, which establishes point d).

To prove points a), b) and c) we will argue (as Osborne, 1993) that, in every equilibrium with $k \geq 3$, any constituency cannot be larger than $\frac{2}{k}$. Assume, without loss of generality, that there is an equilibrium strategy profile with $k \geq 3$, such that the left constituency of y^j is larger than $\frac{1}{k}$. By d) we know that in every equilibrium with $k \geq 3$ each candidate gets a vote share of $\frac{1}{k}$. It follows that at least two candidates are located at y^j. Then candidate $i \in N$ with $y_i = y^j$ can deviate to $y^j - \varepsilon$ for $\varepsilon > 0$ small enough and i) get a vote share larger than $\frac{1}{k}$, ii) reduce the vote share of the other candidate(s) located at y^j, iii) reduce the vote share(s) of the candidate(s) located at y^{j-1} if $j \geq 2$, and iv) leave the vote shares of all other candidates unchanged. Hence such a deviation makes candidate $i \in N$ get a vote share strictly larger than the vote share of any other candidate: the probability that she will get the office under this AVR is at least $\frac{1}{2} > \frac{1}{k}$. That is, every left (right) constituency cannot be larger than $\frac{1}{k}$ and, hence, every constituency must be at most as large as $\frac{2}{k}$.

Point a) clearly follows from d) and the above finding. If $n(y^j) > 2$ and the constituency of y^j is at most as large as $\frac{2}{k}$ then each candidate who locates at y^j gets a vote share strictly smaller than $\frac{1}{k}$, contradicting d). Point c) also follows from the above finding. If $n(y^j) = 2$ then d) suggests that the constituency of y^j is equal to $\frac{2}{k}$. In the previous paragraph we have argued that the left (right) constituency of y^j is at most as large as $\frac{1}{k}$. Hence, the left and the right constituency of y^j are each equal to $\frac{1}{k}$.

Finally, to prove point b), consider without loss of generality that there is an equilibrium strategy profile with $k \geq 3$ such that $n(y^1) = 1$. Point d) implies that in such strategy profiles each active candidate gets a vote share equal to $\frac{1}{k}$. If the candidate who is located at y^1 deviates to $\frac{y^1+y^2}{2}$ she will increase her vote share, reduce the vote shares of the candidates located at y^2, and leave the vote shares of all other candidates unchanged. Hence, this candidate will get a vote share strictly larger than the vote share of any other candidate: the probability that she will get the office under this AVR is at least $\frac{1}{2} > \frac{1}{k}$ for $k \geq 3$. Hence, in every equilibrium with $k \geq 3$ it must be the case that $n(y^1) > 1$. Point a) and this last finding are sufficient to establish that $n(y^1) = n(y^r) = 2$. □

Proof of Lemma 3. The four conditions of Lemma 2 are identical to the four conditions of Lemma 1 in Osborne (1993) despite the fact that the arguments that were needed for their proof are distinct (due to the differences between plurality and the AVR that we consider here). Hence, the arguments of Lemma 2 of Osborne (1993) directly apply here and are sufficient to establish validity of this Lemma. □

Proof of Lemma 4. The reasons which make this Lemma hold are obvious. The AVR that we defined assigns the same election probability ($\frac{1}{2}$) to each of the two active candidates when there are only two active candidates. So if both player 1 and player 2 believe that no other player will enter the race they are indifferent among all available locations since any possible $(y_1, y_2) \in [0,1]^2$ guarantees to each of them a strictly positive vote share and hence an expected payoff equal to $\frac{1}{2}$. □

Proof of Lemma 5. If an equilibrium such that only two candidates enter exists, then it should be such that the two active candidates –say, player 1 and player 2– locate in a manner that discourages any other potential candidate to enter: $y_1 \leq y_2$ are such that if candidate $i = 3$ decides to enter at any $y_3 \in [0,1]$ then her vote share will be either i) strictly smaller than the vote share of candidate 1 and the vote share of candidate 2 or ii) equal to the vote share of one of the two active candidates and strictly smaller than the vote share of the remaining active candidate (in all other cases player $i \geq 3$ would have a positive election probability under the AVR and would prefer to enter the race).

If $y_1 \leq y_2$ and $F(\frac{y_1+y_2}{2}) \leq \frac{1}{3}$ then for $y_3 = y_2$ the vote share of the third candidate coincides with the vote share of the second candidate, exceeds the vote share of the first candidate if $F(\frac{y_1+y_2}{2}) < \frac{1}{3}$ and coincides with the vote share of the first candidate if $F(\frac{y_1+y_2}{2}) = \frac{1}{3}$. In other words, the third candidate will be elected with positive probability if she enters at y_2, and, hence, such a pair (y_1, y_2) cannot characterize an equilibrium with two entrants. A similar reasoning rules out the case in which $y_1 \leq y_2$ and $F(\frac{y_1+y_2}{2}) \geq \frac{2}{3}$. That is, an equilibrium with two entrants must be such that $F(\frac{y_1+y_2}{2}) \in (\frac{1}{3}, \frac{2}{3})$.

If $y_1 \leq y_2$ and $2F(y_1) < F(\frac{y_1+y_2}{2})$ then i) $y_1 < y_2$ and ii) $\lim_{y_3 \to y_1^+}[F(\frac{y_3+y_2}{2}) - F(\frac{y_3+y_1}{2})] = F(\frac{y_1+y_2}{2}) - F(y_1) > F(y_1)$. That is, if the third potential candidate enters arbitrarily close and to the right of y_1, it secures a vote share strictly larger than the vote share of the first active candidate and hence gets elected with a positive probability. A similar reasoning rules out possibility of $2F(y_1) > F(\frac{y_1+y_2}{2})$ and of $2[1 - F(y_2)] \neq 1 - F(\frac{y_1+y_2}{2})$. Therefore, if a strategy profile $(y_1, y_2, Out, Out, ..., Out)$ such that $y_1 \leq y_2$ is an equilibrium of the game, then $(y_1, y_2) \in \mathcal{Y}$. □

Proof of Lemma 6. By Lemma 5.2 of Rubinchik and Weber (2007) we know that for any unimodal and log-concave[18] F there exists a unique $(\dot{y}_1, \dot{y}_2) \in [0,1]^2$ which satisfies all three conditions that characterize elements of set $\mathcal{Y}$. □

Proof of Lemma 7. Notice that for any $(y_1, y_2, Out, Out, ..., Out)$ such that $(y_1, y_2) \in [0,1]^2$ it is straightforward that candidates 1 and 2 have no incentives to deviate. According to AVR each gets the office with a probability equal to $\frac{1}{2}$ (Lemma 4). That is, to show that the proposed strategy profile is actually an equilibrium we need to show that no third candidate has any incentives to enter. Consider that $i = 3$ enters in $y_3 < \dot{y}_1$ ($y_3 > \dot{y}_2$). In this case conditions (A), (B) and (C) which characterize the elements of set $\mathcal{Y}$ suggest that the third candidate gets a vote share which is strictly smaller than the vote share of each of the first two candidates and, thus, the AVR assigns her the office with zero probability, rendering such a deviation not profitable. If $i = 3$ enters in $y_3 = \dot{y}_1$ ($y_3 = \dot{y}_2$) then condition (C) (along with the fact that conditions (A) and (B) imply that $\dot{y}_1 \neq \dot{y}_2$) suggests that $i = 3$ ties with one candidate in the last position and gets a vote share strictly lower than the remaining active candidate; the AVR assigns her the office with probability zero and, hence, this is not a profitable deviation. Finally, consider that $i = 3$ enters in $y_3 \in (\dot{y}_1, \dot{y}_2)$. Notice that condition (A) suggests that $\lim_{y_3 \to \dot{y}_1^+}[F(\frac{y_3+\dot{y}_2}{2}) - F(\frac{y_3+\dot{y}_1}{2})] = F(\dot{y}_1)$. Moreover, $\frac{F(\frac{y_3+\dot{y}_2}{2}) - F(\frac{y_3+\dot{y}_1}{2})}{F(\frac{y_3+\dot{y}_1}{2})}$ (the vote share of the third candidate over the vote share of the first candidate) is decreasing in $y_3 \in (\dot{y}_1, \dot{y}_2)$: $\partial(\frac{F(\frac{y_3+\dot{y}_2}{2}) - F(\frac{y_3+\dot{y}_1}{2})}{F(\frac{y_3+\dot{y}_1}{2})})/\partial y_3 < 0$ if and only if $\frac{f(\frac{y_3+\dot{y}_1}{2})}{F(\frac{y_3+\dot{y}_1}{2})} > \frac{f(\frac{y_3+\dot{y}_2}{2})}{F(\frac{y_3+\dot{y}_2}{2})}$ which always holds due to strict log-concavity of F. That is, the third candidate always gets a vote share smaller than the vote share of the first candidate when she locates at $y_3 \in (\dot{y}_1, \dot{y}_2)$. One can trivially show that the vote share of the third candidate is strictly smaller than the vote share of the second candidate as well when $y_3 \in (\dot{y}_1, \dot{y}_2)$.

Hence, if the first two candidates locate at this unique pair of locations, then no other potential candidate has any individual incentive to deviate and enter the electoral race. That is, $(\dot{y}_1, \dot{y}_2, Out, Out, ..., Out)$ where $(\dot{y}_1, \dot{y}_2) \in [0,1]^2$ such that $\dot{y}_1 \leq \dot{y}_2$ is the unique

[18] They actually consider that F satisfies the *gradually escalating median (GEM)* property which is a weaker requirement compared to the log-concavity of F that we consider here (see Haimanko, Le Breton and Weber, 2005)

pair of strategies which satisfies the three conditions which characterize the elements of set $\mathcal{Y}$, is the essentially unique equilibrium of the game with $k = 2$. □

Proof of Proposition 1. Proposition 1 follows directly from the above Lemmata. □

Proofs of the results presented in Subsection 4.2

Proof of Lemma 8 and of Lemma 9. The arguments that support these two lemmata are similar to the ones used in Lemmata 1 and 2 and are not replicated here. □

Proof of Lemma 10. First of all notice that in equilibrium the set of entrants must be finite. Indeed, given that, in equilibrium, i) every entrant is elected with positive probability, and ii) an entrant is elected with positive probability only if she advances to the lottery stage along with a finite number of other candidates, it follows that, in equilibrium, the number of entrants must be finite.[19] Now consider a strategy profile with a finite number of entrants which satisfies the conditions of Lemma 9. Then, since $n(y^1) = 2$ and the left and the right constituency of y^1 are equal, it follows that $\frac{y^1+y^2}{2} - y^1 = y^1 \Rightarrow y^2 = 3y^1$ and that $y^1 = \frac{1}{k}$. If $n(y^2) = 1$, then the candidate located at y^2 gets a higher vote share than any of the candidates in y^1, so it cannot be true that the vote share of each candidate is equal to $\frac{1}{k}$. Hence, it must be the case that $n(y^2) = 2$. Hence, if a non-entrant decides to deviate and enter at $y \in (y^1, y^2)$ she receives a vote share equal to $\frac{1}{k}$, each of the candidates located at y^1 and y^2 receives strictly less than $\frac{1}{k}$, and any candidate located elsewhere receives a vote share equal to $\frac{1}{k}$. Hence, the entrant is elected with a positive probability, and, therefore, the posited profile cannot be an equilibrium. □

Proof of Lemma 11. Assume that $n(y^j) \geq 2$ for some j and, without loss of generality, that the left constituency of y^j is at least as large as its right constituency. Then, a non-entrant can deviate and enter at $y^j - \varepsilon$ for some $\varepsilon > 0$ sufficiently small, and get a larger vote share than the vote share of each candidate that is located at y^j. Given that the overall number of entrants is $h + 1$ and the new entrant has a larger vote share then two other entrants, it follows that she is elected with positive probability, and hence the posited profile cannot be an equilibrium.If in an equilibrium there exists y^j with a left constituency that is strictly larger than its right constituency, then, since $n(y^j) = 1$, an non-entrant can deviate and enter at $y^j - \varepsilon$ for some $\varepsilon > 0$ sufficiently small, and get a larger vote share than the vote share of the candidate that is located at y^j, and be elected with positive probability. Since $n(y^1) = 1$ and the left and the right constituency of y^1 are equal, it follows that $\frac{y^1+y^2}{2} - y^1 = y^1 \Rightarrow y^2 = 3y^1$ and, hence, the left constituency of y^2 is equal to $y^2 - \frac{y^1+y^2}{2} = 3y^1 - 2y^1 = y^1$, that is, to the left and to the right constituency of y^1. Since the left and the right constituency of y^1 are equal and $n(y^1) = n(y^2) = 1$ the

[19] Observe, though, that, since the set of players is countable, it is possible to have strategy profiles with infinite entrants such that all of them receive a positive vote share (e.g. assume that each candidate $i \in N$ enters at $y_i = \frac{1}{i}$)).

two candidates that are located at these positions get the same vote share. We can apply the same reasoning sequentially to all occupied locations and establish that all candidates get the same vote share, $\frac{1}{k}$. □

Proof of Proposition 4. Since in every equilibrium with $k = h$ we have that i) $n(y^j) = 1$ for all j, ii) the left constituency of y^j is equal to the right constituency of y^j for all j, and iii) the vote share of each entrant is $\frac{1}{h}$, it follows that the occupied locations are such that $(y^1, y^2, ..., y^r) = (\frac{1}{2h}, \frac{3}{2h}, ..., \frac{2h-1}{2h})$. So if an equilibrium exists, then this equilibrium is unique –up to a permutation in players' identities– and it is such that $(y_1, y_2, ..., y_h, y_{h+1}, ...) = (\frac{1}{2h}, \frac{3}{2h}, ..., \frac{2h-1}{2h}, Out, ...)$. To see why this strategy profile is an equilibrium, notice that none of the entrants has any incentive to deviate since by doing that she does not increase the probability that she is elected, and none of the non-entrants has any incentive to enter at any location. Indeed, if a non-entrant, i, deviates and enters at a point at which no other player is located, she gets a vote share equal to $\frac{1}{2h}$ which is strictly smaller than the vote share that any of the original h entrants gets and, hence, she is elected with zero probability. If she enters at the same location as some other candidate then $\#H_i > h$, while there is at least one other candidate, i', with $\#H_{i'} \leq h$. That is, player i is elected with probability zero. Hence, a non-entrant has no incentive to deviate and enter at any location, and the posited profile is indeed an equilibrium. □