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Abstract

This paper studies electoral competition between two non-ideological parties when voters are free to trade votes for money. We find that allowing for vote trading has significant policy consequences, even if trade does not actually take place in equilibrium. In particular, the parties' equilibrium platforms are found to converge (hence, there is no reason for vote trading) to the ideal policy of the mid-range voter, instead of converging to the peak of the median voter (as they do when vote trading is forbidden). That is, a market for votes may not change the outcome only by redistributing the political power among voters when the parties' policy proposals are fixed (e.g., Casella et al. 2012, etc.), but also by acting as an *invisible hand*—modifying parties' incentives when platform choice is endogenous.

Declarations

The authors have no relevant financial or non-financial interests to disclose.

1 Introduction

The analysis in this paper blends two key strands of political economics literature: electoral competition and vote trading. These strands are closely related but have never been combined as they appear to deal with diverse issues. Models of electoral competition have produced a variety of results on how parties choose policy platforms under

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a variety of electoral rules, but—to our knowledge—there is none that allows voters to exchange votes for money. On the other hand, models of vote trading have extensively studied how voters' cardinal preferences determine their vote-market actions, but they consider policy platforms as fixed and hence overlook parties' reactions.[1] In a more general setting where parties select strategically their policy platforms before voters engage in vote trading, rational parties should anticipate the effects of vote markets on the vote distribution and design their policy platforms accordingly. For instance, they could aim at accommodating the preferences of voters who care the most about the implemented policy—as these individuals are potential vote buyers and may end up after vote trading with an increased number of votes—rather than attracting as many supporters as possible. Hence, without ignoring the various criticisms of vote trading,[2] we believe that from a neutral perspective it is interesting to examine the policy implications of vote markets when the parties' policy platforms are endogenous.

We investigate the effects of vote trading on voters' and parties' behavior in a two-party power-sharing system where the implemented policy is a compromise between the competing platforms, with a party's weight on the implemented policy being equal to its vote share.[3] The study of this electoral rule simplifies our analysis and makes it relevant to societies where proportional representation is employed to a significant extent. For example, many countries are parliamentary democracies, and policies represent a

[1]See, for instance, Philipson and Snyder (1996), Casella et al. (2012), Casella and Turban (2014), Casella et al. (2014), Xefteris and Ziros (2017, 2018), Tsakas et al. (2021), Casella and Macé (2021), Casella and Sanchez (2022), among others.

[2]The main criticisms stem from the fact that vote trading affects the payoffs of all voters. This externality complicates the existence of equilibrium and, moreover, leads to ambiguous results concerning social welfare. For an account of the criticisms, one is referred to the papers cited above, which study vote trading with exogenously fixed policy platforms.

[3]Similar approaches are found in many other works. See, for instance, Lijphart (1984), Ortuño-Ortín (1997), Grossman and Helpman (1999), Llavador (2006), Merrill and Adams (2007), De Sinopoli and Iannantuoni (2007), Saporiti (2014), Matakos et al. (2016), among others.

settlement among competing parties. We stress though—and explain at the end of the paper—that the insights provided by the analysis do not depend crucially on this specific system and extend to alternative ones (e.g., simple majority rule).

Concerning our findings, we demonstrate that for any pair of distinct party platforms and for every generic profile of voters' preferences, there exists an equilibrium where all individuals choose to buy or sell votes. In this equilibrium, bidding for purchase of votes comes only from two voters—the extreme (or strongest) supporter of each party—and all other individuals sell their votes. The party that is supported by the voter who is the most concerned about the outcome of the election receives the most votes. Rational vote-share maximizing parties which expect this behavior, do not have incentives to try to be appealing to as many voters as possible (as when vote trading is not allowed), but only to be preferred by the voter who cares most about the electoral outcome. These incentives lead them to choose platforms that converge to the ideal policy of the mid-range voter; that is, the voter who has equal distance from the two extreme supporters.[4] Platform convergence is explained as follows: any deviation from the ideal policy of the mid-range voter towards the one of an extreme voter makes the other extreme voter to be the most concerned about the electoral outcome, and hence leads to an increase in the vote-share of the party that still proposes the ideal policy of the mid-range voter.

It follows that voters do not trade votes for money in equilibrium. However, this is not due to voters' ethical or democratic concerns. On the contrary, all voters are willing to trade if there is any profit in doing so, but platform convergence annihilates the incentives. Yet, allowing for vote trading is consequential. Vote markets are shown to

[4]In light of the equilibrium features, our work is associated with Hirata and Kamada (2020), which studies a two-party election where a party's winning probability depends on the contributions it raises.

be able to affect policy also by acting as an invisible hand: when vote trading is allowed, parties choose the ideal policy of the mid-range voter, and this (generically) differs from the equilibrium outcome when vote trading is not allowed (i.e., the ideal policy of the median voter). While this is reminiscent of electoral competition with costly voting—in equilibrium the two candidates propose the same policy and there is zero turnout (Ledyard 1984)—we note that in that case the outcome is utilitarian, while here it is not. That is, vote markets are neither compatible with majoritarian principles (when the ideal policy of the mid-range voter differs from the one of the median voter), nor they align with utilitarian ones (when the ideal policy of the mid-range voter differs from the utilitarian one).

In what follows, we present the model (Section 2) and the formal results (Section 3).

2 Model

We consider a society of $n > 2$ voters and two parties, $\mathcal{A}$ and $\mathcal{B}$. The two parties simultaneously choose their platforms (α and β respectively) in the policy space $Y = [0, 1]$. Parties are non-ideological and aim at maximizing their vote shares. The vote shares of the two parties are $v_\alpha \in [0, 1]$ and $v_\beta = 1 - v_\alpha$ respectively. Under power sharing, the implemented policy z is a linear combination of party platforms with weights being equal to the vote shares, $z = v_\alpha \alpha + (1 - v_\alpha)\beta$.

Each voter has one vote and one unit of money. Voters' ideal policies are distributed along $Y = [0, 1]$ and each individual i is characterized by her distinct ideal policy y_i. An individual's utility depends on the implemented policy z and is given by

$$u_i = -(y_i - z)^2 + m_i,$$

where $m_i \geq 0$ is the amount of money she ends up with after vote trading.[5,6] All information is publicly known—that is, there is no uncertainty about voters' preferences.

The timing of the game is as follows: First, parties choose their platforms; next, individuals choose their vote-trading actions and to which party to designate their votes, if they have any; finally, the payoffs of all individuals are computed.[7]

Vote trading is modeled as a strategic market game (originating in Shapley and Shubik 1977). There is a trading post where individuals can offer their whole votes for sale, $q_i \in \{0, 1\}$, or place monetary bids for purchase of votes, $b_i \in [0, 1]$; with the restriction that an individual is not allowed to be active on both sides of the market (i.e., $b_i q_i = 0$).

Given a strategy profile, let B and Q denote aggregate bids and offers of all voters. When $BQ = 0$, no trade takes place and hence the allocation of votes and money is $(x_i, m_i) = (1, 1)$ for all voters. For a strategy profile that results in $BQ > 0$, the price of a vote is $p = \frac{B}{Q}$ and the amounts of votes and money of individual i are

$$(x_i, m_i) = \begin{cases} (1 + \frac{b_i}{p}, 1 - b_i) & \text{if } b_i > 0,\ q_i = 0, \\ (0, 1 + p) & \text{if } b_i = 0,\ q_i = 1, \\ (1, 1) & \text{if } b_i = 0,\ q_i = 0. \end{cases}$$

[5] Considering that a voter's utility declines with the distance between her ideal policy and the implemented one is essential for our main result about platform convergence. This is also the case in standard electoral competition models without vote trading; hence, such a modeling allows for a direct comparison with the results when vote trading is not allowed, which is the main inquiry of this paper.

[6] Most assumptions employed (e.g., two parties, quasilinear preferences) are standard in the vote-trading literature (e.g., Casella et al. 2012; Casella et al. 2014; Xefteris and Ziros 2017) and provide a convenient way for solving the problem of nonexistence of equilibrium in a market for votes. This literature also abstracts from several empirically relevant factors in real-world settings (e.g., elite influence, lobbies), which future research should consider.

[7] While abstention is not allowed in our model, our main results are robust to considering voluntary participation.

According to this allocation rule, votes offered for sale are distributed among buyers in proportion to their bids, whereas sellers receive p units of money. Votes are perfectly divisible and hence a buyer might end up having a non-integer number of votes. This is perfectly legitimate in our framework as all that matters is the share and not the actual number of votes that a party receives.

Note that when parties choose identical platforms, the number of votes one has does not affect the implemented policy and hence there is no scope for trade. For distinct party platforms we define the extreme supporter of each party as the voter with the most intense preferences. Without loss of generality, we assume that if $\alpha < \beta$ the extreme voter of party $\mathcal{A}$ has $y_i = 0$ and the extreme voter of party $\mathcal{B}$ has $y_i = 1$; hence, the mid-range ideology is $y = 1/2$.

We also denote by n_α the number of individuals who haven't sold their votes and vote for $\mathcal{A}$, and we denote by B^α the aggregate bids of these individuals.

Given that we have a two-stage game, an equilibrium is defined as a profile of pure strategies that form a subgame perfect equilibrium (SPE).

3 Results

We begin by examining the behavior of individuals in the second stage of the game. Lemma 1 shows that in any equilibrium with trade only one voter of each party submits a bid for purchase of votes.

Lemma 1 *For any subgame with $\alpha < \beta$, in any equilibrium with trade only two individuals—*

one voter of each party—buy votes.

Proof. First, we argue that any equilibrium with trade must involve buyers who support both parties. Assume to the contrary a profile of strategies with $BQ > 0$ where no voters of party $\mathcal{B}$ place bids for purchase of votes. Then a buyer who votes for $\mathcal{A}$ will always deviate by reducing her bid, as such a deviation increases her available money without affecting the vote share of party $\mathcal{A}$. With similar arguments we can establish that there is no equilibrium with trade if there are no buyers who vote for $\mathcal{A}$. Hence, any equilibrium with trade must involve buyers who vote for both parties.

Next, we show that no equilibrium with trade involves two or more buyers who vote for the same party. Consider an individual who votes for party $\mathcal{A}$ and chooses $b_i^\alpha > 0$ in some equilibrium. This individual faces the problem

$$\max_{b_i^\alpha \in [0,1]} u_i = -(y_i - z)^2 + 1 - b_i^\alpha,$$

where $z = \frac{1}{n}(n_\alpha + \frac{B^\alpha}{p})\alpha + (1 - \frac{1}{n}(n_\alpha + \frac{B^\alpha}{p}))\beta$,

which is well-behaved in $b_i^\alpha \in [0, 1]$.[8] Solving the problem and rearranging we derive that in an interior solution we have

$$y_i = \frac{1 + 2z\frac{\partial z}{\partial b_i^\alpha}}{2\frac{\partial z}{\partial b_i^\alpha}}. \tag{1}$$

[8] That is, if at least one other individual sells her vote, the utility function is well-defined, differentiable, and strictly concave in $[0, 1]$.

Similarly, for an individual who votes for party $\mathcal{B}$ and chooses $b_i^\beta > 0$ in some equilibrium, we derive that in an interior solution we have

$$y_i = \frac{1 + 2z\frac{\partial z}{\partial b_i^\beta}}{2\frac{\partial z}{\partial b_i^\beta}}. \tag{2}$$

Assume that there exists an equilibrium where two or more voters of $\mathcal{A}$ submit bids for purchase of votes. Then, their equilibrium bids must satisfy expression (1). However, for that to be the case, all these individuals should have the same ideal policy, which contradicts our assumption about distinct ideal policies.

Similarly, there is no equilibrium with two or more buyers who vote for $\mathcal{B}$, as their equilibrium bids must satisfy expression (2), which is impossible given that each voter has a distinct ideal policy. Thus, no equilibrium with trade involves two or more buyers who vote for the same party. ■

We proceed to show that if platforms diverge, apart from the no-trade equilibrium,[9] equilibria involving trade generically exist. Proposition 1 shows that there is always an equilibrium where the two extreme supporters buy votes, and all other individuals offer their votes for sale. For the remainder of this paper, we use the term full-trade equilibrium when we refer to this equilibrium.

Proposition 1 *For any subgame with $\alpha < \beta$ and for any distribution of ideal policies there exists an equilibrium where individuals with $y_i \in (0,1)$ sell their votes and the two individuals with $y_i \in \{0,1\}$ buy votes.*

[9] Choosing not to trade is always a best response of an individual when all other individuals choose not to trade.

Proof. From Lemma 1 we know that in any equilibrium with trade, equilibrium bids satisfy expressions (1) and (2). Let us consider the possibility of a full-trade equilibrium where the two individuals with $y_i = \{0, 1\}$ buy votes and the remaining individuals sell their votes (that is, $Q = n - 2$).

In such a case, the equilibrium bids of the two buyers are

$$\bar{b}^{\alpha} = \frac{2(n-2)}{n} \frac{(\beta-\alpha)(\alpha+\beta(n-1))^2(\beta+\alpha(n-1)-n)}{(2\beta+\alpha(n-2)-(\beta+1)n)^3}, \tag{3}$$

$$\bar{b}^{\beta} = \frac{2(n-2)}{n} \frac{(\alpha-\beta)(\alpha+\beta(n-1))(\beta+\alpha(n-1)-n)^2}{(2\beta+\alpha(n-2)-(\beta+1)n)^3}, \tag{4}$$

which are both positive and budget feasible for $0 \leq \alpha < \beta \leq 1$ and $n > 2$.[10,11] Given the concavity of the maximization problem, neither of them wishes to deviate to any other bidding amount.

Moreover, the extreme voters never deviate to any other strategy. The utility that the voter with $y_i = 0$ derives from selling her vote is $u_i(b_i = 0, q_i = 1) = -\beta^2 + 1 + \frac{1}{n-1}\bar{b}^{\beta}$ and from refraining from vote trading and voting for $\mathcal{A}$ is $u_i(b_i = 0, q_i = 0; \mathcal{A}) = -(-\frac{1}{n}\alpha - \frac{n-1}{n}\beta)^2 + 1$. Substituting for the posited bid of the other buyer, we derive that the voter with $y_i = 0$ prefers refraining from vote trading to selling her vote.[12] We can also establish that she prefers bidding the equilibrium amount $\bar{b}^{\alpha}$ to refraining from vote trading. If $b_i = 0$ was preferable to $\bar{b}^{\alpha}$, there should have been a local minimum in between. But this is not the case because when the other buyer chooses $\bar{b}^{\beta}$, there is a unique bid (i.e., expression (3)) where the derivative of the utility is zero.

[10] See the Appendix.

[11] We notice that the equilibrium bids of the two buyers are increasing in the size of the electorate, but they are not affected by the ideal policies of all other voters. Moreover, one can easily show that $\bar{b}^{\alpha} > \bar{b}^{\beta}$ ($\bar{b}^{\alpha} < \bar{b}^{\beta}$) whenever $\frac{\alpha+\beta}{2} > \frac{1}{2}$ ($\frac{\alpha+\beta}{2} < \frac{1}{2}$).

[12] See the Appendix.

With similar arguments we can show that the extreme voter with $y_i = 1$ never deviates to selling her vote or to refraining from vote trading. Hence, the posited vote-buying strategies of the extreme voters are their unique best responses.

Next, we show what no individual with ideal policy $y_i \in (0, 1)$ places a monetary bid to acquire more votes. Indeed, given that the voter with $y_i = 0$ buys votes, no other individual who votes for $\mathcal{A}$ is bidding a positive amount as expression (1) cannot be satisfied for any $y_i > 0$. Similarly, given that the voter with $y_i = 1$ buys votes, no other individual who votes for $\mathcal{B}$ submits a positive bid as expression (2) cannot be satisfied for any $y_i < 1$.

Finally, we show that if the extreme voters use the posited vote-buying strategies and all others are expected to sell, then an individual with $y_i \in (0, 1)$ prefers selling her vote to refraining from vote trading. With such a profile of expected behaviors in place, the utility that an individual with $y_i \in (0, 1)$ derives from selling her vote is $u_i(b_i = 0, q_i = 1) = -(y_i - z)^2 + 1 + \frac{1}{n-2}(\bar{b}^\alpha + \bar{b}^\beta)$, where $z = \frac{1}{n}(1 + \frac{\bar{b}^\alpha}{\bar{b}^\alpha + \bar{b}^\beta}(n-2))\alpha + (1 - \frac{1}{n}(1 + \frac{\bar{b}^\alpha}{\bar{b}^\alpha + \bar{b}^\beta}(n-2)))\beta$. On the other hand, the utility that she derives from voting for $\mathcal{A}$ without engaging in vote trading is $u_i(b_i = 0, q_i = 0; \mathcal{A}) = -(y_i - \hat{z})^2 + 1$, where $\hat{z} = \frac{1}{n}(2 + \frac{\bar{b}^\alpha}{\bar{b}^\alpha + \bar{b}^\beta}(n-3))\alpha + (1 - \frac{1}{n}(2 + \frac{\bar{b}^\alpha}{\bar{b}^\alpha + \bar{b}^\beta}(n-3)))\beta$. Substituting for the posited bids of the extreme voters, we derive that an individual with $y_i \in (0, 1)$ prefers selling her vote to refraining from vote trading and voting for $\mathcal{A}$.[13]

With similar arguments we can establish that an individual with $y_i \in (0, 1)$ prefers selling her vote to voting for $\mathcal{B}$ without engaging in vote trading. Thus, all individuals with $y_i \in (0, 1)$ sell their votes.

[13] See the Appendix.

Hence, a full-trade equilibrium always exists when platforms diverge and is characterized by the actions described above. ■

Our results concerning the patterns of vote trading when policy platforms are fixed (i.e., the fact that only the strongest supporter of each party buys votes and all others sell their ballots; and that the party supported by the most concerned voter receives a higher share of votes) align with earlier analyses in alternative institutional contexts. Indeed, Casella and Turban (2014) derive sufficient conditions for the existence of a similar equilibrium in a majoritarian system, employing an ex ante competitive equilibrium (a solution notion introduced in Casella et al. 2012). Xefteris and Ziros (2018) also describe a similar equilibrium considering that the policy outcome is probabilistic (i.e., the platform of each party is implemented with a probability equal to its vote share).[14]

Let us note that for certain parameter values, apart from the full-trade equilibrium, there are also other equilibria where some individuals trade while others prefer not to engage in vote trading. In all partial-trade equilibria (if they exist), two individuals buy votes, some individuals sell their votes, and the remaining individuals choose not to trade. In such a case, more than two voters get to vote: the two buyers (each casting more than one vote) and the non-traders (each casting exactly one vote).

[14] In this paper the policy outcome is deterministic (i.e., a weighted average of the parties' platforms, with weights being equal to their vote shares). When voters' utility functions are strictly concave (as they are here), then the probabilistic setup is not equivalent to the deterministic one: in the former case, the relationship between a voter's utility and the vote share of her preferred party is linear, while in the latter, the relationship is strictly concave, making the problem distinctly more complicated. For this reason, we cannot simply refer to earlier arguments to establish Proposition 1.

Arguably, the current deterministic setup, where policy is a compromise of the two platforms and not the product of a "random dictatorship", is a better assumption in many ways and it aligns with how several papers in the literature treat policy formation in the presence of divergent platforms (e.g., Ortuño-Ortín 1997; Merrill and Adams 2007; Matakos et al. 2016). Moreover, by employing this assumption, the current paper, beyond its main contribution (i.e., to show how allowing for vote trading affects policy outcomes when platforms are endogenous), makes a secondary point: it establishes that earlier results provided in probabilistic settings, are also valid in more standard settings of deterministic policy formation.

However, partial-trade equilibria are non-generic as they exist only for specific profiles of voters' preferences. A partial-trade equilibrium requires that the ideal policies of all non-traders belong to specific intervals of the policy space. If, instead, there are no individuals whose ideal policies belong to these intervals, then there are no partial-trade equilibria. The following result presents one class of partial-trade equilibria by considering a simple case where parties choose symmetric platforms and the two buyers are the extreme voters.

Lemma 2 *For any subgame with* $\alpha \in [0, \frac{1}{2})$ *and* $\beta = 1 - \alpha$*, there exists an equilibrium where the two individuals with* $y_i = \{0, 1\}$ *choose* $\bar{b}^{\alpha} = \bar{b}^{\beta} = \frac{(1-2\alpha)(n-2-2k)}{4n}$*,* $n - 2 - 2k$ *individuals sell their votes,* k *individuals refrain from vote trading and vote for party* $\mathcal{A}$*, and* k *individuals refrain from vote trading and vote for party* $\mathcal{B}$*, if there are* k *individuals with* $y_i \in (0, \xi]$ *and* k *individuals with* $y_i \in [1 - \xi, 1)$*, where* $\xi = \frac{(1-2\alpha)(n-1-2k)+2n}{4n(n-1-2k)}$*.*

Proof. In the Appendix.

We note that individuals with $y_i \in (0, \xi]$ or $y_i \in [1 - \xi, 1)$ do not refrain from vote trading in all equilibria of the game. What we prove is that if these individuals are not expected to trade—which leads to lower equilibrium bids by the two buyers—they have no incentives to deviate to selling. Hence, equilibria with trade and more than two voters are due to coordination issues.[15] Let us also note that $\frac{\partial \xi}{\partial \alpha} = -\frac{1}{2n} < 0$; that is, as party platforms converge, the intervals of ideal policies of non-traders shrink.

[15] Future research could consider additional factors (e.g., concave utilities in money or uncertainty regarding the ideal policies of other individuals) that might provide further justification for a larger number of voters.

Next, we examine the behavior of the two parties in the first stage of the game, and we present the main result of the paper.

Proposition 2 *There exists a SPE where party platforms converge to the mid-range ideology. Moreover, this SPE is unique among those that involve a full-trade equilibrium in every subgame with distinct platforms.*

Proof. First, we show that the profile $\alpha = \beta = 1/2$ is an equilibrium. If $\alpha = \beta = 1/2$, there is no vote trading and a party's expected vote share is equal to 1/2. Suppose that party $\mathcal{A}$ deviates to a platform $\hat{\alpha} \in [0, 1/2)$. Such a deviation induces vote trading and, in the full-trade equilibrium, expressions (3) and (4) with $\beta = 1/2$ yield

$$\bar{b}^{\alpha} = \frac{2(n-2)}{n} \frac{(\frac{1}{2} - \hat{\alpha})(\hat{\alpha} + \frac{1}{2}(n-1))^2(\frac{1}{2} + \hat{\alpha}(n-1) - n)}{(1 + \hat{\alpha}(n-2) - \frac{3}{2}n)^3},$$

$$\bar{b}^{\beta} = \frac{2(n-2)}{n} \frac{(\hat{\alpha} - \frac{1}{2})(\hat{\alpha} + \frac{1}{2}(n-1))(\frac{1}{2} + \hat{\alpha}(n-1) - n)^2}{(1 + \hat{\alpha}(n-2) - \frac{3}{2}n)^3},$$

for which we can easily derive $\frac{\bar{b}^{\beta}}{\bar{b}^{\alpha}} = \frac{-\frac{1}{2}+\hat{\alpha}+(1-\hat{\alpha})n}{-\frac{1}{2}+\hat{\alpha}+\frac{1}{2}n} > 1$ for $\hat{\alpha} \in [0, 1/2)$ and $n > 2$. That is, the buyer with $y_i = 1$ submits a greater bid than the buyer with $y_i = 0$, and hence the vote share of party $\mathcal{A}$ will be less than 1/2. In other words, party $\mathcal{A}$ cannot increase her vote share by choosing a different platform. Similarly, there is no profitable deviation for party $\mathcal{B}$.

Next, we show that no profile involving $\alpha = \beta \neq 1/2$ is an equilibrium. If $\alpha = \beta \neq 1/2$, there is no vote trading, and the two parties have equal expected vote shares. In such an eventuality, there is always a profitable deviation. For example, let $\alpha = \beta < 1/2$ and suppose that party $\mathcal{B}$ deviates to platform $\hat{\beta} = 1/2$. Such a deviation induces vote trading with $\bar{b}^{\beta} > \bar{b}^{\alpha}$ and hence the vote share of party $\mathcal{B}$ increases.

Finally, we show that no profile involving $\alpha \neq \beta$ is an equilibrium. If $\alpha \neq \beta$, there is vote trading leading either to a tie or to a majority winner. In such an eventuality, using similar arguments as before, we can always find a profitable deviation. For example, if $\alpha \neq 1/2$ and $v_\alpha \leq 1/2$, then party $\mathcal{A}$ can increase its vote share by deviating to $\hat{\alpha} = 1/2$. ■

The reason for platform convergence to the mid-range ideology—and not to the median one—lies in the fact that, when vote trading is allowed, parties compete for the bids of the two extreme voters and not for the votes of centrist citizens as in the standard Downsian model. Indeed, as we saw in the proof of Proposition 1, the electoral outcome does not depend on the preferences of the voters with non-extreme ideal policies.[16] Moreover, the bid of an extreme voter is increasing in the utility difference between the two platforms. Since the voters' utilities are concave in the implemented policy, the most concerned voter is always the one farthest away from the midpoint between the two platforms, leading parties to converge exactly to the midpoint between the ideal policies of the two extreme voters.

Considering the discussion following Proposition 1 about similar results with respect to the number of vote traders, we finally note that our main result (Proposition 2) remains relevant in many more settings than the one employed by this paper. That is, the novel finding that parties are expected to converge to the mid-range ideology when vote trading is allowed, does not hinge on the specific voting system and trading mechanism, but should continue to hold in alternative settings as well. The reason why we opted for a

[16] This is very reminiscent of actual behaviors in legislatures. Indeed, empirical research provides evidence that the relevant party ideology in legislatures is more extreme than the ideology of the median party legislator (see, for instance, Grofman et al. 2002).

power-sharing rule and a strategic market game to formally solve this model is twofold: (i) this setting is accepted in the literature as a relevant one when thinking about the policy consequences of vote trading (see, for instance, Casella and Macé 2021; Tsakas et al. 2021; Xefteris and Ziros 2017), and, perhaps more importantly, (ii) this set of assumptions allows us to provide a compact, yet, complete formal argument establishing the main result.

Indeed, the current modeling approach allows us to proceed with a standard SPE analysis, while if we assumed, instead, a majority rule and an ex ante competitive market for votes—like Casella and Turban (2014)—we could still arrive to a similar conclusion, but we would first need to define a novel solution notion which would combine elements of SPE (i.e., parties should compete in the first stage taking into account their beliefs regarding their competitor's behavior and the subsequent behavior of the voters), and ex ante competitive equilibrium (for the voters' "subgames").

Appendix

Calculations showing that the equilibrium bids $\bar{b}^{\alpha}$, $\bar{b}^{\beta}$ are positive and budget feasible:

In expression (3) the numerator is negative because $\beta + \alpha(n-1) - n < 0$ and the remaining terms are positive for $0 \leq \alpha < \beta \leq 1$ and $n > 2$. The denominator is negative because $2\beta + \alpha(n-2) - (\beta+1)n = \beta(2-n) + n(\alpha-1) - 2\alpha < 0$ for $0 \leq \alpha < \beta \leq 1$ and $n > 2$. Hence, $\bar{b}^{\alpha}$ is positive. Moreover $\bar{b}^{\alpha} < 1$, as each term $\frac{(n-2)}{n}$, $\frac{(\alpha+\beta(n-1))^2}{(2\beta+\alpha(n-2)-(\beta+1)n)^2}$, $\frac{2(\beta-\alpha))(\beta+\alpha(n-1)-n)}{2\beta+\alpha(n-2)-(\beta+1)n}$ is less than one for $0 \leq \alpha < \beta \leq 1$ and $n > 2$.

In expression (4) the numerator is negative because $(\alpha - \beta) < 0$ and the remaining terms are positive for $0 \leq \alpha < \beta \leq 1$ and $n > 2$. The denominator is negative because $2\beta + \alpha(n-2) - (\beta+1)n = \beta(2-n) + n(\alpha-1) - 2\alpha < 0$ for $0 \leq \alpha < \beta \leq 1$ and $n > 2$. Hence, $\bar{b}^{\beta}$ is positive. Moreover $\bar{b}^{\beta} < 1$, as each term $\frac{(n-2)}{n}$, $\frac{2(\alpha-\beta)(\alpha+\beta(n-1))}{2\beta+\alpha(n-2)-(\beta+1)n}$, $\frac{(\beta+\alpha(n-1)-n)^2}{(2\beta+\alpha(n-2)-(\beta+1)n)^2}$ is less than one for $0 \leq \alpha < \beta \leq 1$ and $n > 2$.

Calculations showing that the voter with $y_i = 0$ prefers refraining from vote trading to selling her vote:

In the full-trade equilibrium, substituting for the posited bid of the other buyer, the utility of the voter with $y_i = 0$ from refraining from vote trading and voting for $\mathcal{A}$ is $u_i(b_i = 0, q_i = 0; \mathcal{A}) = -(-\frac{1}{n}\alpha - \frac{n-1}{n}\beta)^2 + 1$ and from selling her vote is $u_i(b_i = 0, q_i = 1) = -\beta^2 + 1 + \frac{2(n-2)}{n(n-1)} \frac{(\alpha-\beta)(\alpha+\beta(n-1))(\beta+\alpha(n-1)-n)^2}{(2\beta+\alpha(n-2)-(\beta+1)n)^3}$. Their difference is $u_i(b_i = 0, q_i = 0; \mathcal{A}) - u_i(b_i = 0, q_i = 1) = \frac{1}{n^2}(\beta-\alpha)\left(\alpha + \beta(2n-1) + \frac{2n(n-2)(\alpha+\beta(n-1))(\beta+\alpha(n-1)-n)^2}{(n-1)(2\beta+\alpha(n-2)-(1+\beta)n)^3}\right) =$

$\frac{1}{n^2}(\beta-\alpha)\left(\beta n + (\alpha+\beta(n-1))(1 + \frac{2n(n-2)(\beta+\alpha(n-1)-n)^2}{(n-1)(2\beta+\alpha(n-2)-(1+\beta)n)^3})\right) > 0$ for $0 \leq \alpha < \beta \leq 1$ and $n > 2$, because the term $\frac{n(n-2)(\beta+\alpha(n-1)-n)^2}{(n-1)(2\beta+\alpha(n-2)-(1+\beta)n)^3}$ is negative and its absolute value is less than one. Thus, the absolute value of the term $1 + \frac{2n(n-2)(\beta+\alpha(n-1)-n)^2}{(n-1)(2\beta+\alpha(n-2)-(1+\beta)n)^3}$ is less than one for $0 \leq \alpha < \beta \leq 1$ and $n > 2$, which yields $\beta n + (\alpha + \beta(n-1))(1 + \frac{2n(n-2)(\beta+\alpha(n-1)-n)^2}{(n-1)(2\beta+\alpha(n-2)-(1+\beta)n)^3}) > 0$.

Calculations showing that an individual with $y_i \in (0,1)$ prefers selling her vote to refraining from vote trading and just voting for $\mathcal{A}$:

In the full-trade equilibrium, substituting for the posited bids of the extreme voters, the utility of an individual with $y_i \in (0,1)$ from selling her vote is $u_i(b_i = 0, q_i = 1) =$

$$\frac{-2(\alpha-\beta)^3+(\alpha-\beta)^2(-3+2\alpha-2\beta-4y_i(y_i-1))n+2(\alpha-\beta)((\alpha-1)y_i(2y_i-1)+\beta(-2+\alpha+(3-2y_i)y_i))n^2-(\beta(y_i-1)+y_i-\alpha y_i)^2n^3}{n((\beta-\alpha)(n-2)+n)^2}$$

and from refraining from vote trading and voting for $\mathcal{A}$ is

$$u_i(b_i = 0, q_i = 0; \mathcal{A}) = -\frac{((\alpha-\beta)^2-(\alpha-\beta)(\alpha-2+2y_i)n+(\beta+(\alpha-\beta-1)y_i)n^2)^2}{n^2((\beta-\alpha)(n-2)+n)^2}.$$

Their difference is $u_i(b_i = 0, q_i = 1) - u_i(b_i = 0, q_i = 0; \mathcal{A}) =$

$\frac{(\alpha-\beta)(\beta+\alpha(n-1)-n)\left(-(\alpha-\beta)^2+(\alpha-\beta)(-1+\alpha+4y_i)n+2(1+\beta-\alpha)y_in^2\right)}{n^2((\beta-\alpha)(n-2)+n)^2} > 0$ for $y_i \in (0,1)$, $0 \leq \alpha < \beta \leq 1$ and $n > 2$, because the product $(\alpha - \beta)(\beta + \alpha(n-1) - n)$ is positive and the term $(-(\alpha-\beta)^2 + (\alpha-\beta)(-1+\alpha+4y_i)n + 2(1+\beta-\alpha)y_in^2)$, which can be written as $(\beta-\alpha)((\alpha-\beta) + (1-\alpha)n) + (\beta-\alpha)(2n-4)y_in + 2y_in^2$, is also positive.

Proof of Lemma 2. Considering that $\beta = 1 - \alpha$ and $Q = n - 2 - 2k$, expressions (1), (2) yield that the equilibrium bids of the two individuals with $y_i = \{0, 1\}$ are $\bar{b}^\alpha = \bar{b}^\beta = \frac{(1-2\alpha)(n-2-2k)}{4n}$, which are positive and budget feasible for $\alpha \in [0, \frac{1}{2})$, $n > 2$ and $k < \frac{n-2}{2}$. Given the concavity of the maximization problem, neither of the two extreme voters wishes to deviate to any other bid.

Moreover, the two voters with $y_i = \{0, 1\}$ never deviate to any other strategy. Given the posited strategies of the other players, the utility that the voter with $y_i = 0$ derives from playing $\bar{b}^\alpha$ is $u_i(b_i = \bar{b}^\alpha, q_i = 0; \mathcal{A}) = -(-\frac{1}{2})^2 + 1 - \frac{(1-2\alpha)(n-2-2k)}{4n}$ and from refraining from vote trading and voting for $\mathcal{A}$ is $u_i(b_i = 0, q_i = 0; \mathcal{A}) = -(-\frac{k+1}{n}\alpha - (\frac{n-k-1}{n})(1-\alpha))^2 + 1$. Their difference is $u_i(b_i = \bar{b}^\alpha, q_i = 0; \mathcal{A}) - u_i(b_i = 0, q_i = 0; \mathcal{A}) = \frac{(1-2\alpha)(n-2k-2)}{2n^2}(n - k - 1 - n\alpha + 2\alpha + 2k\alpha) > 0$ for $\alpha \in [0, \frac{1}{2})$, $n > 2$ and $k < \frac{n-2}{2}$. Furthermore, the utility that the voter with $y_i = 0$ derives from selling her vote is $u_i(b_i = 0, q_i = 1) = -(-\frac{k}{n}\alpha - (\frac{n-k}{n})(1-\alpha))^2 + 1 + \frac{1}{n-1-2k}\frac{(1-2\alpha)(n-2-2k)}{4n}$ and hence $u_i(b_i = \bar{b}^\alpha, q_i = 0; \mathcal{A}) - u_i(b_i = 0, q_i = 1) = \frac{(1-2\alpha)(n-2k)^2}{4n^2(n-2k-1)}(2n - 2k - 2n\alpha - 1 + 2\alpha + 4k\alpha) > 0$ for $\alpha \in [0, \frac{1}{2})$, $n > 2$ and $k < \frac{n-2}{2}$.

With similar arguments we can show that the voter with $y_i = 1$ never deviates to selling her vote or to refraining from vote trading. Hence, the posited bids of the two extreme voters are their unique best responses.

Next, we show what no individual with ideal policy $y_i \in (0,1)$ places a monetary bid to acquire more votes. Given the posited bids of the voters with $y_i = \{0,1\}$, there is no positive bid that satisfies expression (1) for an individual with $y_i > 0$ who votes for party $\mathcal{A}$. Similarly, there is no positive bid that satisfies expression (2) for an individual with $y_i < 1$ who votes for party $\mathcal{B}$.

Next, we consider an individual with $y_i \in (0,1)$ who sells her vote in this partial-trade profile of strategies and all others expect it. Substituting for the posited strategies of the other players, the utility from selling her vote is $u_i(b_i = 0, q_i = 1) = -(y_i - \frac{1}{2})^2 + 1 + \frac{(1-2\alpha)}{2n}$ and from voting for party $\mathcal{A}$ without engaging in vote trading is $u_i(b_i = 0, q_i = 0; \mathcal{A}) = -(y_i - \hat{z})^2 + 1$, where $\hat{z} = \frac{1}{n}(2 + k + \frac{1}{2}(n - 3 - 2k))\alpha + (1 - \frac{1}{n}(2 + k + \frac{1}{2}(n - 3 - 2k)))(1 - \alpha)$ $= \frac{1}{2n}(n + 2\alpha - 1)$. Their difference is $u_i(b_i = 0, q_i = 1) - u_i(b_i = 0, q_i = 0; \mathcal{A}) = \frac{(1-2\alpha)(4ny_i - 2\alpha + 1)}{4n^2} > 0$ for $\alpha \in [0, \frac{1}{2})$ and $n > 2$; that is, she has no incentives to deviate to voting for party $\mathcal{A}$ without engaging in vote trading. With similar arguments we can establish that an individual with $y_i \in (0,1)$ who sells in a partial-trade profile of strategies will not deviate to voting for party $\mathcal{B}$ without engaging in vote trading.

Consider now an individual who refrains from vote trading and just votes for party $\mathcal{A}$ in this partial-trade profile of strategies and all others expect it. Substituting for the posited strategies of the other players, her utility from refraining from vote trading is $u_i(b_i = 0, q_i = 0; \mathcal{A}) = -(y_i - \frac{1}{2})^2 + 1$ and from selling her vote is $u_i(b_i = 0, q_i = 1) = -(y_i - \tilde{z})^2 + 1 + \frac{1}{n-1-2k}\frac{(1-2\alpha)(n-2-2k)}{2n}$ where $\tilde{z} = \frac{1}{n}(k + \frac{1}{2}(n - 1 - 2k))\alpha + (1 - \frac{1}{n}(k +$

$\frac{1}{2}(n-1-2k)))(1-\alpha) = \frac{1}{2n}(n-2\alpha+1)$. This individual will not deviate to selling her vote if $u_i(b_i = 0, q_i = 0; \mathcal{A}) \geq u_i(b_i = 0, q_i = 1) \Rightarrow y_i \leq \xi = \frac{(1-2\alpha)(n-1-2k)+2n}{4n(n-1-2k)}$. That is, an individual with $y_i \in (0, \xi]$ who refrains from vote trading and votes for $\mathcal{A}$ in this partial-trade profile of strategies will not deviate to selling her vote.

Similarly, an individual with ideal policy $y_i \in [1-\xi, 1)$ who refrains from vote trading and just votes for $\mathcal{B}$ in this partial-trade profile of strategies will not deviate to selling her vote. ■

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