

Credit Risk Measurement in Financial Institutions: Going Beyond Regulatory Compliance

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Abstract

Capital adequacy is an important factor considered by financial institutions when they formulate their lending policy and balance sheet growth strategy. The majority of financial institutions employ the Standardised Approach for calculating their credit risk capital requirements as they cannot meet the stringent criteria stipulated in Basel II (and Basel III) and qualify for the more advanced approaches. The Standardised Approach lacks the necessary risk sensitivity and the resulting regulatory capital requirements serve as a very crude proxy of the actual credit risk taken. Strategic decision making based on this approach, often provides institutions with a perverse incentive for pursuing (a) collateral-driven lending policies rather than focusing on obligor financial standing and repayment ability; (b) balance sheet short-term growth strategies where excess liquidity takes the form of high-yield government bond investments. This paper presents two simplified credit risk models that are not data demanding and, by addressing the very weaknesses of the Standardised Approach, more informative in measuring the possible future loss impact of credit risky business or investment decisions. It provides a comparative analysis of the presented models with empirical results suggesting that financial institutions would need to do more than simply maintaining compliance with the minimum regulatory capital requirements.

Keywords: Capital allocation, Capital requirements regulations, Credit risk measurement, Eurozone banking crisis, Value-at-Risk (VaR).

1. Introduction

One of the lessons learned from the recent Eurozone banking crisis is that minimum credit risk capital requirements and the regulatory approach on which their calculation relies can influence or even dictate lending policy or investment decisions. The majority of the financial institutions in the

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Eurozone region cannot meet the stringent data and risk management practice requirements of the advanced approaches, namely the Internal Ratings Based (IRB) Approaches stipulated in Basel II (and Basel III). As a result they employ the Standardised Approach for measuring credit risks and making credit risky strategic decisions. The Standardised Approach essentially relies on the weighting of credit risk exposures using some risk weight values preset by regulators by exposure class. The risk weight values make sense for large and well diversified asset portfolios, but their meaningfulness for smaller or concentrated portfolios remains questionable particularly during times of financial stress.

The use of Standardised Approach alone for quantifying loan book credit risks and other counterparty credit risks, and reliance on this approach for managing such risks and making business or investment decisions can be misleading. It is not risk sensitive enough and often provides institutions with a perverse incentive for pursuing (a) collateral-driven lending policies rather than focusing on obligor financial standing and repayment ability; (b) balance sheet short-term growth strategies where excess liquidity takes the form of government bond investments offering high yields yet carrying a zero risk weight. It can be argued that the high non-performing loan (NPL) ratios in customer loan portfolios and disproportionate credit risk positions in high-yield government bonds (rated below investment grade) in bank balance sheets, observed during the Eurozone crisis, are partly attributable to the use of the Standardised Approach.

This paper proposes the use of two simplified models that are not data demanding and can be used by smaller sized banks not able to qualify for IRB. It provides a detailed theoretical review of the said models followed by their implementation, illustration of their use, and comparative analysis of their output. It argues that financial institutions should not merely limit their efforts to regulatory compliance; they would need to go beyond that if they want to ensure that they have enough cushion for absorbing potential credit risk losses.

The proposed models are applied to a commercial loan portfolio and a sovereign bond holding. The findings from this illustration suggest that the Standardised Approach grossly underestimates the possible future losses from certain credit risk exposures. The proposed credit risk models on the other hand are much more risk sensitive, promote credit risk diversification, and prepare risk takers to be more sufficiently capitalized ahead of periods of financial turmoil¹.

¹ That said, the regulatory capital requirements for credit portfolios with a low to moderate credit risk profile are expected to be higher than the capital really needed, which goes back to the lack of risk sensitivity from which the Standardised Approach suffers.

2. Standardised approach

The Basel II (and Basel III) regulatory framework² stipulate that banks should set aside sufficient capital as a cushion against potential unexpected credit risk losses that may arise over the next year, or a longer horizon, this way protecting bank depositors from a default event. Regulators have prescribed the approaches that institutions should employ for calculating their minimum regulatory capital requirements. There are three progressively more advanced approaches, namely the Standardised Approach, the Foundation IRB Approach, and the Advanced IRB Approach. The qualitative and quantitative criteria that institutions must fulfill for the IRB approaches are pretty stringent and not many can qualify for employing such approaches. Consequently most banks must employ the Standardised Approach.

In principle, the Standardised Approach is straight forward. It considers each and every asset on or off the balance sheet, and assigns an appropriate credit risk weight value based on the regulatory exposure class. The minimum credit risk regulatory capital requirements is the product of the total risk weighted assets and 8%³. There is a price to be paid though for over-simplifying things. The following sections depict three pitfalls underlying the Standardised Approach. For the same reasons discussed below, this approach may underestimate or even overestimate in some cases the potential credit risk losses depending on the portfolio credit quality, collateral coverage, and degree of diversification.

2.1 Pitfall one

The type of eligible collateral is the sole risk driver in secured lending under the Standardised Approach. Although risk weight values vary by obligor type (retail, corporate, bank, sovereign) and collateral type (residential real estate, commercial real estate, lien funds, other financial collateral) the latter always comes first and that could pose a problem in some cases.

For example, a Euro 100,000 unsecured loan is risk weighted by 75% if granted to a private individual, and 100% if granted to an unrated (by a recognised credit ratings agency) corporate. However, if the loan is fully secured with an eligible residential real estate, the risk weight would become 35% regardless whether this credit is granted to a private

² See Basel Committee (2004), Capital Requirements Regulation (CRR) under EU Regulation (2013).

³ Excluding bank-specific Pillar 2 capital requirements, and additional capital buffers introduced in EU Regulation (2013).

individual or a corporate and regardless whether the real estate sector shows signs of overheating. Similarly if the loan is secured with a commercial property the risk weight would become 50% again regardless of the obligor type and the state of the economy. This weakness of the Standardised Approach, to be driven solely by the collateral type, creates a perverse incentive for lending on the basis of collateral regardless of the financial standing and repayment ability of the obligor. The source of repayment is implicitly taken to be the proceeds from the forced sale value of the collateral which in practice can be significantly discounted during an economic crisis.

2.2 Pitfall two

The risk weight values considered by the Standardised Approach are static. This means the credit risk taken by the financial institution from granting a credit facility is by design perceived to be stationary. Risk weight values are insensitive to changes in the idiosyncratic or systematic risk factors that could adversely impact obligor credit quality over time.

For example, if an unrated corporate obligor's financial standing deteriorates because of increased competition in the market place and a drop in sales turnover, the risk weight shall remain 100% despite a likely increase in the actual default probability.

Another representative example is the zero risk weight applicable on EU sovereign credit risks; this risk weight value makes sense when a sovereign enjoys a strong external credit rating and government bond yield spreads or credit default swap spreads are at sane levels, but it raises questions when a country shows signs of economic decline or is one step before insolvency.

2.3 Pitfall three

The risk weight values stipulated in the Standardised Approach apply on a one-size-fits-all basis. The same set of risk weight values apply to concentrated and well diversified credit portfolios, small and large financial institutions, small and large economies. If the risk weight values employed by the Standardised Approach are meaningful for large and well diversified loan portfolios or developed economies that typically rely on a wide range of economic activities, how can they apply at the same time to loan portfolios with high concentrations, or small economies relying on a handful of industry sectors? On the other hand, if the risk weight values are calibrated to apply to concentrated loan portfolios, are they not unnecessarily conservative, from a capital requirement

standpoint, for diversified loan portfolios, misleading financial institutions' strategic decision making? Moreover, the majority of obligor-corporates in small or medium sized economies, where most Standardised Approach financial institutions operate, are not externally rated. These realities take away further from Standardised Approach's risk sensitivity or, more accurately, add to the lack of it.

2.4 Literature review

Acharya and Steffen (2013) provided empirical evidence that some European banks took advantage of the Standardised Approach risk weighting scheme by concentrating in zero risk-weight sovereign exposures of the Southern European periphery. On this point Acharya (2011) went on to say that the significance of the problem is clear in the Eurozone debt crisis during which the zero capital requirement on sovereign debt, issued by Eurozone countries, was not in line with the assessments of their riskiness at the time.

Acharya et al (2014), using publicly available data, assessed the effectiveness of macro-prudential stress tests and concluded that the average risk weight of European banks appears completely uncorrelated with their actual risk, and that a risk-weight based approach for calculating capital requirements is not sufficient as there is a risk that risk will change⁴; for example the risk of an increase in the credit risk over time of some currently safe asset classes such as residential mortgages or government bonds. They argued that the Standardised Approach risk weights are flawed measures of bank risks as they ignore the sub-additivity feature of portfolio risk and allow for arbitrage, i.e. cherry picking on one or two risky asset classes with a low risk-weight to meet minimum regulatory capital requirements, which does not necessarily reduce economic leverage. In fact the concentration in bank asset portfolios coupled with the underestimation of risk weights inevitably leads to excessive leverage⁵.

Sonali and Amandou (2012) also questioned the credibility of the risk weights arguing that asset-risk measurement should be revised by regulators. Empirical evidence from their work suggests that although

⁴ Also see Engle (2009).

⁵ In response to the recent financial crisis the European regulator, through the CRR, introduced the leverage ratio, being the Tier 1 Own Funds divided by Total Exposures. Unlike the risk weighted assets (which is the denominator in the capital adequacy ratio) Total Exposures do not rely on risk weights or CRR eligible risk mitigation techniques. The idea is to restrict banks from building up excessive leverage while maintaining strong risk-based capital adequacy ratios.

banks with lower risk weighted assets performed better (in terms of stock market returns) during US and European crises, for large banks investors paid less attention to risk weighted assets and rewarded instead better asset quality (lower NPLs). Their findings lead to the conclusion that risk weighted assets do not, in general, predict market measures of banks' riskiness. Evidence from the US in fact indicated that this relationship is negative after the 2008 crisis. According to the authors, this could result from the large increase in the market measures of risk, which reflect the volatility of a bank's stock price, since the 2008 US crisis, while banks have not adjusted their risk weighted assets to account for the increased risk.

3. Purpose and contents

3.1 Purpose

The Standardised Approach suffers from a number of shortcomings as highlighted in Section 2. Financial institutions at the same time must adopt this approach for calculating the minimum credit risk capital requirements under the Pillar 1 of Basel II and Basel III (and CRR in EU). Does this regulatory requirement mean that institutions should be using the same approach for assessing their actual internal or economic capital needs? Not necessarily. Regulatory compliance is one thing, being sufficiently capitalized and able to absorb unexpected credit risk losses that could potentially arise from a specific balance sheet in a given economy is another, one could argue.

The primary purpose of this paper is twofold: (a) to present from first principles a detailed theoretical review of two data efficient and simplified credit risk models that do not suffer from the structural weaknesses of the Standardised Approach, and (b) to implement these two models and provide a comparative analysis of their output. The latter is illustrated by applying the models on a commercial loan portfolio and a sovereign bond holding, and measuring the credit risk and statistical loss profile of these exposures using tail-sensitive risk measures.

The proposed credit risk models have a wide scope of application. They can be used for managing credit risks as well as facilitating business or investment decision making processes, including internal capital adequacy assessment and stress testing, forward looking bad debt provisioning, risk-based credit product pricing, credit risk mitigation and collateral management, and fixed income portfolio management.

3.2 Credit risk modelling

Unlike the Standardised Approach, credit risk models allow for the fundamental credit risk parameters to speak for themselves. The aim herein is to employ risk sensitive models that require the minimum necessary data. To this end, this paper deals with default risk exclusively; other credit risks such as obligor credit rating migration risk or prepayment risk are not accounted for.

Credit default probabilities are treated as exogenous in the said models. Their values can be obtained directly from external credit ratings when such ratings are available. Otherwise they can be estimated from the long run averages of historical frequency transitions of internal credit ratings.

The first of the two models proposed in this paper, termed the Actuarial Model, assumes that credit defaults occur as a Negative Binomial stochastic process or, in its simplified version, a Poisson process. Credit default correlations are handled through the standard deviation of annual default probabilities, designed to be influenced by the systematic risk factors to which obligors in a portfolio are commonly exposed. There are three stages to implementing the model: (a) the portfolio is classified by exposure size so that all obligors of a certain size range are bucketed together in 'bands', (b) the default distribution is calculated for each band, and (c) the bands are aggregated together to obtain the annual loss distribution for the entire portfolio. Several risk measures, including the Value-at-Risk (VaR) and Expected Shortfall, are extracted from the created annual loss distribution. A tail sensitive VaR, based on Extreme Value Theory, is also developed for looking into the far end of the tail of the annual loss distribution.

The second proposed model, termed the Vasicek Single Factor Model, begins with the principle that obligor asset returns are explained by systematic and idiosyncratic risk factors. Asset returns are assumed to follow a Gaussian distribution. Under certain conditions, only the systematic risk factors are taken to contribute to asset return correlations which in turn determine the credit default correlations. Credit default events are conditioned on the state of the economy, causing the systematic credit risk, this way eliminating dependencies and enabling the derivation of the joint default probability in a closed form.

3.3 Contents

The remaining part of the paper is organized as follows: Section 4 provides a comparative analysis of the results produced by the two credit risk models versus the Standardised Approach using data illustrations. Section

5 concludes. The data requirements, notation, model specification and all other technical information appear in the appendices. Appendix A lays down the required data model input and provides a complete theoretical exposition of the Actuarial Model followed by its implementation. Appendix B offers a theoretical exposition of the Vasicek Single Factor Model followed by its implementation.

4. Data illustration: credit risk capital allocation

The Actuarial Model developed in Appendix A and the Vasicek One Factor Model developed in Appendix B find several applications in banking credit risk management as well as fixed income portfolio management. This section illustrates their use in banking credit risk management, more specifically in credit risk measurement and capital allocation which is part of the internal capital adequacy (forward looking) assessment process that banks must conduct annually. Section 4.1 considers a commercial loan portfolio, and Section 4.2 considers a sovereign bond position.

4.1 Commercial loan portfolio

For the purpose of this illustration we consider a commercial loan portfolio with total exposure Euro 100 million net of collateral. The obligors making up this loan portfolio operate in one of six industry sectors of the economy, namely (a) Real Estate, (b) Construction, (c) Wholesale & Retail Trade, (d) Hotels & Restaurants, (e) Manufacturing, (f) Transportation & Communication. The annual default probability values range from 0% to 22% and average at 3%. The annual default probability standard deviation, required by the Actuarial Model, is taken to be equal to the 50%⁶ of the default probability value implied by the obligor credit ratings.

4.1.1 Scenario specification

Three scenarios have been created:

Scenario 1: Assume all the credit facilities in the loan portfolio have a common systematic risk so that an economic shock will impact all obligors in the portfolio to the same extend.

⁶ The 50% is taken from a methodology by Credit Suisse, which proposed the Actuarial Model approach in fixed income portfolio management. See Credit Suisse (1997).

- *Implementation for Actuarial Model:* No sectoral split; assume all obligors operate in the same industry sector. Set the number of sectors $K=1$ in the aggregate credit default annual loss probability generating function specified by (22) or the equivalent probability mass function specified by (23) in Appendix A.
- *Implementation for Vasicek One Factor Model:* Set the loading parameter (correlation) $\rho = 24\%$, which is the maximum possible value as per IRB approach (for Corporates) specified by (40) in Appendix B.

Scenario 2: Account for diversification effects by recognizing that different obligors operate in different industry sectors, i.e. not all obligors will be impacted to the same extent if there is a shock in the economy.

- *Implementation for Actuarial Model:* Account for the sector in which each obligor in the portfolio operates; each exposure must be assigned to one of the six industry sectors. Set the number of sectors $K=6$ in (22).
- *Implementation for Vasicek One Factor Model:* Allow the loading parameter ρ to vary with default probability based on (40). Effectively ρ takes values in the range $[12\%, 24\%]$ depending on each obligor's default probability value.

Scenario 3 - Stress conditions: Same as Scenario 2 only that now three of the six industry sectors (namely, construction, real estate, and wholesale and retail trade) are assumed to have been adversely impacted by an economic downturn. The selected stressed industry sectors represent 76% of the total portfolio net exposure of Euro 100 million.

- *Implementation for Actuarial Model:* Increase the default probability standard deviation for the selected industry sectors to 90% of the default probability value (from 50% assumed in Scenarios 1 and 2 for all industry sectors).
- *Implementation for Vasicek One Factor Model:* For the deteriorated sectors nearly double the ρ value from that used in Scenario 2, using 24% as the maximum. That is, the adjusted loading factor ρ' for the selected sectors is given by $\rho' = \min \{(9/5) \rho, 24\%$ while the ρ values for the remaining sectors remain the same to those used in Scenario 2 (as determined by (40))⁷.

⁷ The choice of scaling value 9/5 is in line to the increase of the default probability volatility from 50% to 90% in the Actuarial Model although this choice does not suggest that the sensitivity of VaR to default probability volatility and ρ is the same. The rationale here is simply to nearly double the parameters in the two models controlling the systematic credit risk as part of a stress scenario exercise. This working assumption could be replaced with a

The following two sub-sections present a summary of the numerical results of the one year annual loss distributions obtained from running the two credit risk models on the commercial loan portfolio data for each scenario, and discuss their economic meaning and implications. For comparison purposes, the capital requirement calculated for this loan portfolio by the Standardised Approach (at 8% minimum capital adequacy ratio) is Euro 14.6 million. This amount holds for all three scenarios since the Standardised Approach cannot capture the risk factors differentiating the three scenarios. The Euro 14.6 million capital requirement calculated by the Standardised Approach should be compared to the 99.9% confidence level one year total *unexpected* loss estimates produced by the two credit risk models.

4.1.2 Actuarial model illustration

The results obtained from the three scenarios are summarized in the table below.

Scenario 1

The Expected Loss is calculated at Euro 3.7 million. This loss amount is considered to be the cost of doing business under normal market conditions and represents an estimate of the total provisions to be taken over the next year⁸. The same amount holds for all three scenarios since the exposure, default probability and recovery rate values used across all three scenarios are the same.

The 99.9% quantile, which serves as the 99.9% one year credit VaR, is Euro 20.5 million (under Scenario 1). One can therefore claim with 99.9% confidence that the maximum total one year loss associated with the credit exposure to the said portfolio is Euro 20.5 million, or 20.5% of the portfolio total net exposure. The total one year unexpected loss is Euro 16.8 million. If a bank allocates credit risk capital Euro 16.8 million it anticipates a default event (capital cushion being lower than actual total credit losses) once every 1000 years. This amount is compared to Euro 14.6 million calculated by the Standardised Approach.

Scenario 2

This illustration accounts for the presence of six different industry sectors associated with obligors in the portfolio. From an economic perspective,

more robust quantitative study linking macro-economic variables (e.g. GDP and unemployment rate) to a measure of systemic credit risk if the relevant historical data are available.

⁸ Assuming no impaired loans exist in the portfolio considered for this illustration.

the credit quality of an obligor in the portfolio is no longer dependent on the whole economy but on a given industry sector in that economy. By accounting for the sectoral diversification effects, the resulting annual loss distribution gains more mass at the centre (the area around Expected Loss) whereas the tail becomes shorter and thinner compared to that resulting from Scenario 1. While the median of the annual loss distribution increased by Euro 0.1 million, the loss quantiles are increasingly lower as one moves from the centre towards the tail (see table below). The 99.9% VaR is Euro 18.6 million compared to Euro 20.5 million in Scenario 1. Thus, sectoral diversification mitigates systematic risk which gives rise to unexpected losses.

The 99.9% one year unexpected loss is Euro 14.9 million compared to Euro 14.6 million calculated by the Standardised Approach. The latter fails by design to capture risk diversification and its mitigating impact on portfolio losses, however this weakness is counterbalanced by the tendency to underestimate individual risks. Essentially, there are two weaknesses with opposite loss implications here explaining the proximity of the Standardised Approach to the Actuarial Model result.

Scenario 3

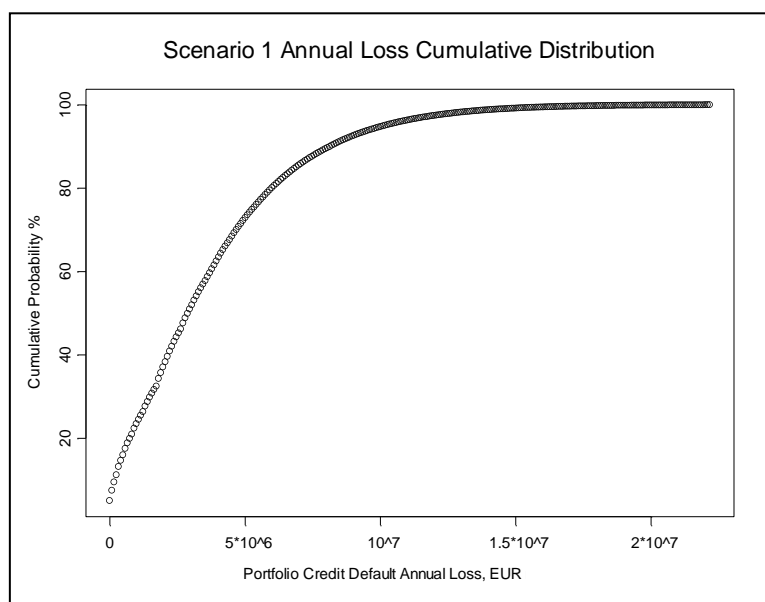
In this illustration, the default probability standard deviation is set at 90% of the default probability value (compared to 50% used in Scenarios 1 and 2) for the obligors operating in the three deteriorating industry sectors. The default probability standard deviation for the obligors in the remaining industry sectors remains set at 50%. The increase in the default probability volatility for the selected three sectors causes a stretching of the tail with the mass around the Expected Loss being rather squashed. Evidently, the quantile loss values from the 95% confidence level and beyond are increasingly larger compared to the Scenario 2 results; the 99.9% VaR, for instance, stands at Euro 22.9 million compared to Euro 18.6 million in Scenario 2. The median of the annual loss distribution is Euro 2.8 million compared to Euro 3.0 million in Scenario 2.

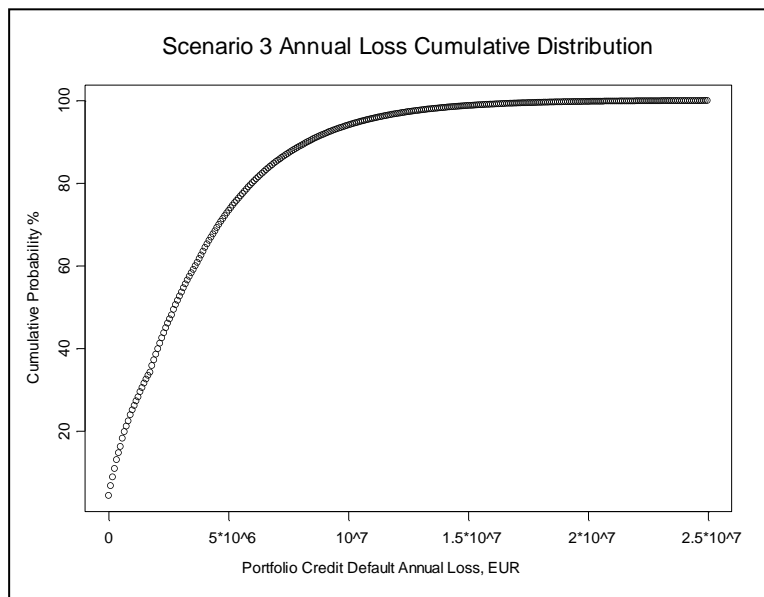
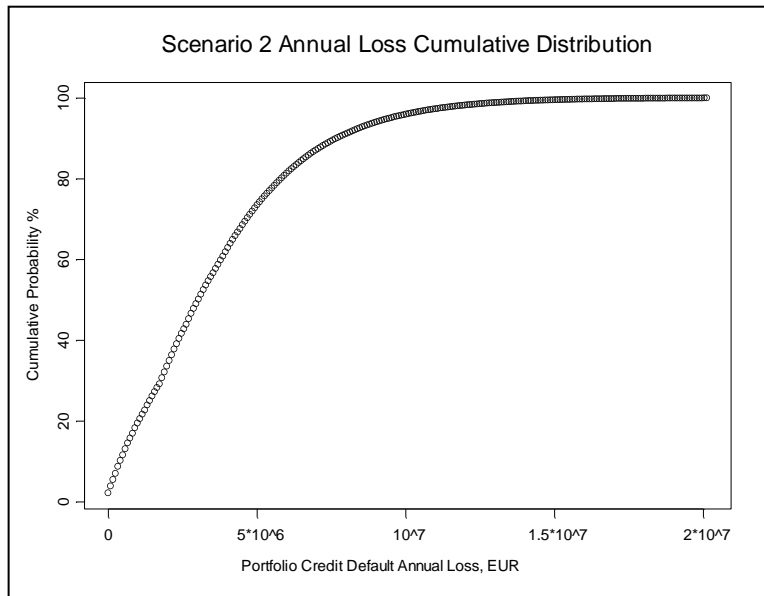
The 99.9% one year unexpected loss is Euro 19.2 million compared to Euro 14.6 million calculated by the Standardised Approach. By design, the Standardised Approach is unable to adjust itself for deteriorating economic conditions.

TABLE 1
Actuarial Model results summary

in Euro million	Portfolio Total Credit Loss		
Annual Loss Distribution Statistic	Scenario 1 (no sectoral split)	Scenario 2 (sectoral diversification)	Scenario 3 (stress conditions)
Mean	3.7	3.7	3.7
Median	2.9	3.0	2.8
75% Quantile	5.3	5.2	5.2
95% Quantile	10.2	9.5	10.6
99% Quantile	14.6	13.4	15.7
99.5% Quantile	16.4	15.0	17.9
99.9% Quantile (VaR)	20.5	18.6	22.9
99.9% Unexpected Loss (=VaR-Mean)	16.8	14.9	19.2

Note: The 99.9% Unexpected Loss model estimates above are compared to the Euro 14.6 million capital requirement produced by the Standardised Approach.





4.1.3 Vasicek one factor model illustration

Although the Vasicek One Factor Model is not directly comparable to the Actuarial Model, they are designed to solve the same problem and the respective results shouldn't be too different. The calculated Expected Loss amount is Euro 3.7 million, i.e. identical to the results presented in Section 4.1.2 since this risk measure is not model dependent. But what about the profile of unexpected losses? The results obtained from the three scenarios are summarized in the table below.

Scenario 1

By comparing the annual loss distributions simulated by the two models under Scenario 1 it is clear that they are very close up to the 75% quantile, however further into the tail results show marked differences. In particular, the Vasicek One Factor model yields significantly larger default annual losses from the 99% quantile and beyond; the 99% quantile annual loss amount is Euro 6.3 million larger than that produced by the Actuarial Model, while the 99.5% quantile annual loss amount is larger by Euro 8.3 million. The 99.9% quantile annual loss (VaR) stands at Euro 33.7 million i.e. Euro 13.2 million larger than the Actuarial Model estimate. From a capital perspective this means the Vasicek One Factor Model is more conservative than the Actuarial Model, not necessarily closer to reality though.

The 99.9% one year unexpected loss is Euro 30.0 million compared to Euro 14.6 million calculated by the Standardised Approach. This is significantly higher than the gap between the Actuarial Model corresponding estimate and the Standardised Approach.

The Expected Shortfall 99.9% VaR provides information on the area of the annual credit loss distribution beyond the 99.9% VaR. It is a linear and conditional risk measure complementary to VaR, quantifying the mean of all large annual loss values exceeding VaR⁹. Two credit risks may have the same loss profile up to the VaR for a given confidence level, but their tail beyond VaR may be significantly different (e.g. the tail of a commercial loan portfolio is typically heavier than that of a retail loan portfolio); in this case the Expected Shortfall VaR values will be different although the VaR values may coincide. With regards to the commercial loan portfolio considered in the illustration, the Expected Shortfall 99.9% VaR amounts to Euro 39.7 million, meaning if total credit losses associated to the said loan portfolio exceed Euro 33.7 million over the next year the excess is expected at Euro 6.0 million. If a bank allocates capital only for unexpected losses up to 99.9% VaR it risks insolvency if such tail credit losses occur over the next year. This is where banks should consider credit transfer techniques such as credit default swaps, or simply accepting the risk and either allocating

⁹ Expected Shortfall VaR can be calculated empirically by $E(X | X > VaR) = \frac{1}{n_{VaR}} \sum_{i=1}^{n_{VaR}} (X_i)$

where random variable X denotes default annual losses and n_{VaR} the number of annual losses exceeding VaR.

Furthermore, $E(X - VaR | X > VaR) = \frac{1}{n_{VaR}} \sum_{i=1}^{n_{VaR}} (X_i - VaR) = E(X | X > VaR) - VaR$ represents the corresponding Expected Excess.

additional internal capital and/or implementing credit risk concentration limits. Doing nothing and hoping that the 'default year' over the next 1000 years is not going to be the next one could be disastrous.

Scenario 2

The 99.9% one year credit VaR is calculated at Euro 24.1 million, and the Expected Shortfall 99.9% VaR at Euro 28.4 million. Compared to Scenario 1 one can observe the impact of diversification effects on unexpected losses which are significantly lower under this scenario, in particular from the 99% quantile and beyond.

It can be said that Scenario 2 lends itself for a more direct comparison between the Vasicek One Factor Model and the Actuarial Model since all model parameters are now utilized. The simulated annual loss distributions are remarkably close up to the 99% quantile. However, the annual loss estimates begin to diverge from the 99.5% quantile onwards; the 99.5% quantile loss estimate produced by the Vasicek Model is Euro 3.1 million larger than that produced by the Actuarial Model, while the 99.9% quantile estimate is Euro 5.5 million larger. It is therefore clear that the Vasicek model is more 'heavy tailed' than the Actuarial model, anticipating larger unexpected losses from future default events over the next year.

Another interesting aspect is the model sensitivity to risk diversification effects. What is clear from Scenario 2 results is that the tail impact of diversification effects, through the recognition of different sectors, in the Actuarial Model is much more mild than the tail impact of diversification effects through the lowering of ρ -values in the Vasicek Model. The 99.9% VaR is representative of this observation; it has decreased by Euro 1.9 million in the annual loss distribution produced by the Actuarial Model (Scenario 1 vs 2) and by Euro 9.6 million in the loss distribution produced by the Vasicek Model (Scenario 1 vs 2).

The 99.9% one year unexpected loss is Euro 20.4 million compared to Euro 14.6 million calculated by the Standardised Approach.

Scenario 3

The 99.9% one year credit VaR is calculated at Euro 31.8 million, and the Expected Shortfall 99.9% VaR at Euro 37.4 million. Compared to Scenario 2, the impact from the deterioration of the three industry sectors is reflected in the unexpected loss amounts which are significantly larger from the 99% quantile and beyond.

Compared to the Actuarial Model results, the Vasicek Model appears to respond much more intensively to systematic risks. By comparing the

quantile annual loss increases (Scenario 2 vs 3) between the two models it is clear that the systematic shock introduced through the higher default probability volatility values in the Actuarial Model has a lower tail impact compared to an equivalent systematic shock introduced through the higher ρ -values in the Vasicek Model. Indicatively, the 99.9% VaR increased by Euro 4.3 million in the case of Actuarial Model (Scenario 2 vs 3), and by Euro 7.7 million in the case of Vasicek Model (Scenario 2 vs 3).

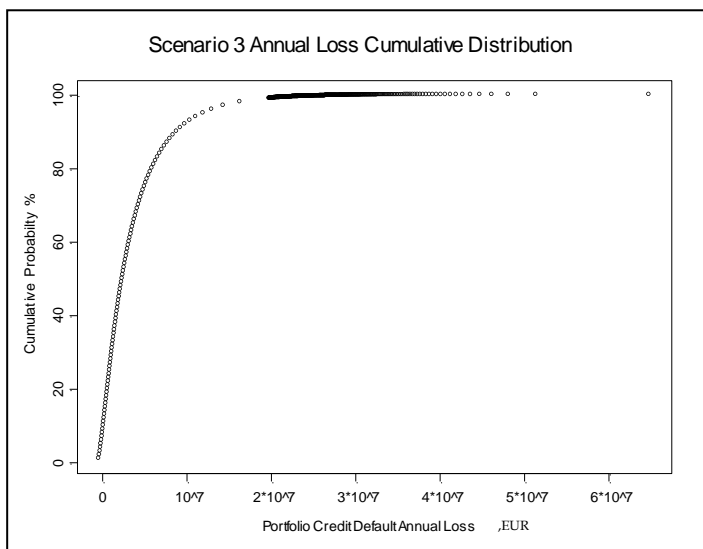
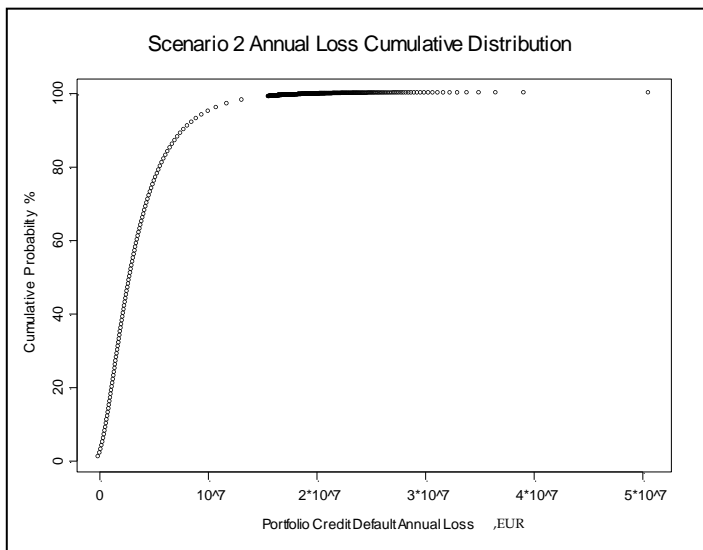
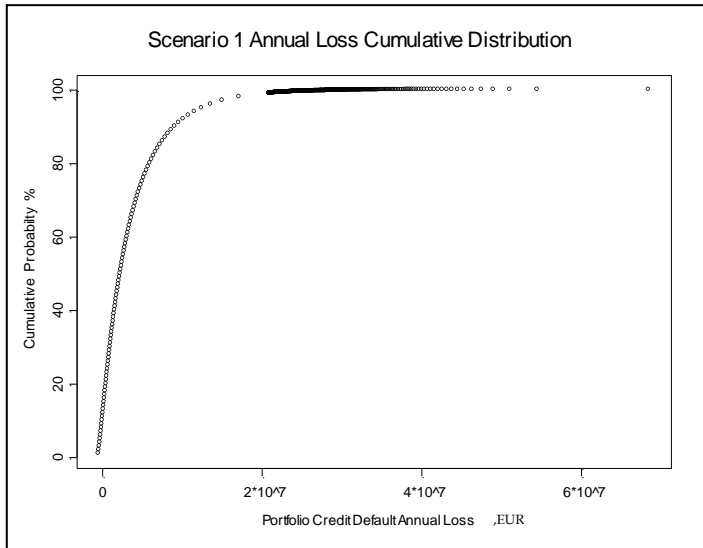
The 99.9% one year unexpected loss is Euro 28.1 million compared to Euro 14.6 million calculated by the Standardised Approach.

TABLE 2

Vasicek One Factor Model results summary

in Euro million	Portfolio Total Credit Loss		
Annual Loss Distribution Statistic	Scenario 1 (no sectoral split)	Scenario 2 (sectoral diversification)	Scenario 3 (stress conditions)
Mean	3.7	3.7	3.7
Median	2.3	2.8	2.4
75% Quantile	5.1	4.9	5.0
95% Quantile	12.5	10.1	12.0
99% Quantile	20.9	15.6	19.8
99.5% Quantile	24.7	18.1	23.4
99.9% Quantile (VaR)	33.7	24.1	31.8
99.9% Unexpected Loss (=VaR-Mean)	30.0	20.4	28.1
Expected Shortfall 99.9% VaR	39.7	28.4	37.4
Expected Excess over 99.9% VaR	6.0	4.2	5.6

Note: The 99.9% Unexpected Loss model estimates above are compared to the Euro 14.6 million capital requirement produced by the Standardised Approach.



4.1.4 Illustrating the use of extreme value theory

Earlier in Section 4.1.2 no Expected Shortfall 99.9% VaR figures were reported on the Actuarial Model due to its inherent shortcomings discussed in the introduction of Section A.4 in Appendix A. Extreme Value Theory (EVT) is used in an attempt to fill in this gap. EVT is introduced in Section A.4.1. A tail-sensitive quantile risk measure is developed in Section A.4.2 and is applied to the credit default annual loss distribution produced by the Actuarial model under Scenario 3.

The beginning of the tail (marked by threshold parameter 'u' in the context of the Generalised Pareto distribution (GPD) specified in (24)) was set at Euro 8 million with 11% of the loss data exceeding u. The GPD was fitted to that 11% largest annual losses. The GPD parameter maximum likelihood estimates are Euro 3.4 million for the scale (β) and -0.071 for the shape (ξ), i.e. the fitted tail resembles an exponential distribution. The table below presents the calculated values of one year EVaR, specified in (28), at 99.9%, 99.97% and 99.98% confidence levels. The 99.9% VaR estimated directly by the Actuarial Model as well as the same three quantiles estimated directly by the Vasicek One Factor Model (without applying EVT) are also presented for comparison.

EVaR is tail sensitive but it does not seem to inflate the capital needs as evidenced by a comparison of the Actuarial Model VaR and EVaR figures at 99.9%.

More importantly, EVaR provides information on a crucial part of the tail where the Actuarial Model alone is not capable to do so. Comparing the Actuarial Model EVaR results to the corresponding VaR figures estimated directly by the Vasicek One Factor Model one confirms the conclusions drawn earlier in Section 4.1.3. Whereas up to around the 95% quantile the two models produce distributions with almost the same statistical profile, thereafter the Vasicek One Factor Model produces more conservative results than the Actuarial model does from a capital perspective. In Section 4.1.3 this observation was said to be partly attributed to the way the Actuarial Model handles correlations rendering it less sensitive to systematic shocks. It can also be explained by the phase type distribution issue discussed in the introduction of Section A.4.

From an economic capital perspective if a bank holding the commercial loan portfolio does operate at 99.9% VaR under Scenario 3 and wishes to reduce credit risk appetite in an attempt to achieve an AAA external rating (i.e. capitalised at 99.98% VaR), it would mean raising another Euro 3.7 million capital under the Actuarial Model or Euro 8.4 million under the Vasicek One Factor Model. In the light of these data, an informed decision

can be taken as to whether some of the risk should be transferred to a third party if possible, or accept the risk and raise capital, or mitigate the risk gradually by implementing a more tight concentration risk limit policy.

TABLE 3

Scenario 3 extreme loss quantiles summary

in Euro million	Portfolio Total Credit Loss in Scenario 3 (stress conditions)		
Annual Loss Distribution Statistic	Actuarial-EVT Model EVaR	Actuarial Model VaR	Vasicek One Factor Model VaR
99.9% Quantile (VaR)	21.5	22.9	31.8
99.97% Quantile	24.2	<i>na</i>	38.1
99.98% Quantile	25.2	<i>na</i>	40.2

Note: na=cannot be available.

4.2 Sovereign Bond

The proposed two credit risk models are applied to a Greek government bond holding. The period April to May 2010 is considered for the purpose of this illustration. This period is characterised by a high volatility of the Greek government bond market prices. It is noted that Greek government bond prices declined from 98 on 1st April 2010 to 82 by the end of that month. During the weekend 1st-2nd May 2010 Eurozone country finance ministers, ECB and IMF approved and announced a Euro 110 billion rescue package for Greece which was not perceived as sufficient by the capital markets with the Greek government bond market prices sliding down to below 80 by the first week of May 2010. On 8th-9th May 2010 Eurozone country finance ministers, ECB and IMF approved and announced an additional Euro 750 billion rescue package which was deemed by the markets as sufficient, quickly pushing the Greek government bond market prices to 95 for the remaining three weeks of May 2010.

Should a risk manager measure the credit riskiness of such an investment based on the Standardised Approach the assessed risk would be zero. So if a bank held that investment during that period there would be a zero capital requirement based on the Standardised Approach.

The following sections present the data, model results and capital needs for a Euro 100 million credit risk position in Greek government bonds.

4.2.1 Risk parameters and working assumptions

Historical sovereign defaults are rare and country specific¹⁰. As a result, external credit ratings on sovereigns tend to be less-timely and sticky¹¹. Here we use instead the annual default probabilities estimated by Vrugt (2011) based on Greek government bond market prices in the period April-May 2010. This methodology is based on the idea to (a) obtain the theoretical bond price by discounting the probability-weighted cash-flows at risk free rates, and (b) estimate the default probabilities by minimising the sum of squared pricing errors where an error is defined as the difference between a bond market price and the corresponding theoretical price. The estimated annual default probabilities presented in Table 4 are based on the weekly closing market prices, for the period April-May 2010, for outstanding bonds maturing in three to ten years.

TABLE 4

Greek gov. annual default probability weekly estimates

Date	Greek government Default Probability (PD) estimates
April 2010	
Week 1	6%
Week 2	5%
Week 3	13%
Week 4	24%
May 2010	
Week 1	40%
Week 2	9%
Week 3	12%
Week 4	10%
Two month period (April-May 2010)	
Average	12.0%
Standard Deviation	9.4%

¹⁰ See Moody's (2012) for more information on past sovereign default cases.

¹¹ For example, the Moody's long term credit rating for Greece was A2 until December 2010 when revised downwards to A3. Moody's subsequently downgraded Greece to Ba1 in March 2011, then to B1 in May 2011. Three more downgrades followed in the period from June 2011 to March 2012 when Greek debt was restructured (restricted default event).

The loss given default (LGD) estimate of 50.7% provided by Moody's (2012) has been adopted for the purpose of this illustration. This estimate is considered to be robust since it is close in value to the LGDs produced by other agencies or independent modelling approaches, see for instance Standard & Poor's (2011) and Vrugt (2011). Finally, the remaining maturity is taken to be three years while the loading parameter ρ (specified by Eq. (40) in Appendix B) is set at 24% in the absence of diversification benefits.

4.2.2 Vasicek One Factor Model results

The results are summarised in Table 5. Since the default probabilities vary in time (as per Section 4.2.1) so does the measured credit risk and capital needed. The worst week from the 8-week period considered is week 1 in May 2010 when Greek bond market prices fell below 80, default probability estimates reached 40% and Expected Shortfall 99.9% VaR soared to Euro 49.8 million or approximately 50% of the exposure. One could argue that the latter risk measure could serve as a leading indicator of what was about to happen nearly two years later when in March 2012 Greece restructured its debt to private sector investors with a 53.5% bond face amount write-down and implied net present value losses of approximately 75%.

4.2.3 Actuarial Model results

The default probability standard deviation is set at 50% of the default probability weekly estimates tabulated in Section 4.2.1. The results are summarised in Table 6; the right-most column of the table is based on the default probability average and standard deviation obtained from historical data in the period April - May 2010.

The annual loss distribution appears to be rather flat in the centre and, separately, in the tail areas because the Actuarial Model requires sufficiently many exposures as it relies on arithmetic (or discrete) loss severity distributions¹². The model typically requires 100 exposures at minimum in order to produce an informative annual loss profile.

¹² This observation also explains the marginal differences on the Expected Loss values coming from the two models.

TABLE 5
Vasicek One Factor Model results summary

in Euro million	Portfolio Total Credit Loss								Two month period Historical Average PD
	April 2010				May 2010				
	Week 1	Week 2	Week 3	Week 4	Week 1	Week 2	Week 3	Week 4	
Annual Loss Distribution Statistic									
Mean	3.1	2.6	6.6	12.2	20.4	4.6	6.1	5.1	6.1
Median	1.7	1.3	4.8	10.5	19.6	3.0	4.3	3.4	4.3
75% Quantile	4.3	3.5	9.5	17.4	27.7	6.5	8.8	7.3	8.8
95% Quantile	11.1	9.7	19.5	29.1	38.6	15.0	18.5	16.2	18.5
99% Quantile	18.4	16.5	28.0	37.1	44.5	23.1	26.9	24.5	26.9
99.5% Quantile	21.5	19.5	31.2	39.8	46.2	26.3	30.1	27.7	30.1
99.9% Quantile (VaR)	28.2	26.1	37.5	44.5	48.8	33.0	36.5	34.3	36.5
99.9% Unexpected Loss (=VaR-Mean)	25.2	23.6	30.9	32.2	28.4	28.4	30.4	29.2	30.4
Expected Shortfall 99.9% VaR	32.1	30.1	40.6	46.5	49.8	36.6	39.7	37.7	39.7
Expected Excess over 99.9% VaR	3.9	3.9	3.1	2.0	1.0	3.5	3.2	3.4	3.2

Note: The 99.9% Unexpected Loss model estimates above are compared to the zero capital requirement as per Standardised Approach.

TABLE 6
Actuarial Model results summary

in Euro million	Portfolio Total Credit Loss								Two month period Historical Average PD and Std Dev* PD
	April 2010				May 2010				
Annual Loss Distribution Statistic	Week 1	Week 2	Week 3	Week 4	Week 1	Week 2	Week 3	Week 4	
Mean	3.0	3.0	7.0	12.0	20.0	5.0	6.0	5.0	6.0
Median	0	0	0	0	0	0	0	0	0
75% Quantile	0	0	0	0	51.0	0	0	0	0
95% Quantile	50.0	0	51.0	51.0	51.0	50.0	51.0	50.0	51.0
99% Quantile	51.0	51.0	51.0	51.0	51.0	51.0	51.0	51.0	51.0
99.5% Quantile	51.0	51.0	51.0	51.0	51.0	51.0	51.0	51.0	51.0
99.9% Quantile (VaR)	51.0	51.0	51.0	51.0	51.0	51.0	51.0	51.0	51.0
99.9% Unexpected Loss (=VaR-Mean)	48.0	48.0	44.0	39.0	31.0	46.0	45.0	46.0	45.0

Note: The 99.9% Unexpected Loss model estimates above are compared to the zero capital requirement as per Standardised Approach.

* 'Std Dev' stands for Standard Deviation.

5. Concluding remarks

The proposed two credit risk models clearly outperform the Standardised Approach. Due to structural weaknesses the latter is not sufficiently risk sensitive; it grossly underestimates the loss profile for certain credit risk exposures and fails capturing portfolio diversification effects. With respect to the Standardised Approach three pitfalls described in Section 2:

- Pitfall One: When it comes to secured lending the two credit risk models handle collateral and default probability separately through

dedicated risk parameters, whereas collateral overrides obligor repayment ability in the regulatory risk weights assumed by the Standardised Approach.

- Pitfall Two: The two credit risk models are sensitive to changes in the credit risk profile due to obligor idiosyncratic or systematic risk factors since the key risk parameters (namely, default probability, default probability volatility or correlation, and loss-given-default) can be adjusted accordingly. On the other hand the risk weight values assumed by the Standardised Approach are static.
- Pitfall Three: The two credit risk models can take into account obligor or portfolio or country specific features. Whereas the Standardised Approach risk weight values are universal, failing to capture stylized facts characterizing smaller-sized portfolios, banks or economies such as the unavailability of external credit ratings for corporate obligors.

In comparing the two credit risk models, the data illustration results herein suggest that:

- The Vasicek Model is more ‘heavy tailed’ than the Actuarial Model, anticipating larger unexpected losses from future default events.
- The EVaR measure is tail sensitive yet not overly conservative from a capital perspective. It provides information on a crucial part of the loss tail where the Actuarial Model alone cannot.
- The Actuarial Model requires sufficiently many exposures to work, typically 100 at minimum.
- The Actuarial Model is less sensitive to portfolio diversification effects than the Vasicek Model.
- The Actuarial Model is less responsive to deteriorating economic conditions or systematic risk shocks, producing more stable results on the other hand, than the Vasicek Model.

The credit risk models presented herein have a wider scope of application than credit risk measurement and capital allocation. They can be applied into risk-based credit product pricing, credit risk mitigation and collateral management, future NPL provisioning, fixed income portfolio management, and overall strategic planning.

Appendix A: Actuarial Model

A.1 Actuarial Model overview

In order to implement the model, the following information must be specified for each obligor exposure in a credit portfolio:

- Credit exposure Euro amount, including non-funded credit facilities¹³
- Exposure segment¹⁴;
- Collateral Euro amount¹⁵;
- Credit rating¹⁶ and implied annual default probability value;
- Industry sector name or country name or other systematic risk factors with a significant bearing on obligors' ability in servicing their loans or other credit obligations.

A general note on terminology that applies to both credit risk models proposed herein: the terms 'obligor', 'exposure', 'obligor exposure' are used interchangeably throughout the paper. In cases where there are multiple exposures to the same obligor in the portfolio, such exposures are treated separately by the models, with distinct collateral and default probability values. The models can also be implemented on a single obligor basis, by aggregating the exposures of the same obligor and imposing additional assumptions with regards to collateral coverage.

The one year credit default loss distribution is driven by two stochastic processes in the Actuarial Model:

- Annual default frequency;
- Credit loss severity.

The credit loss severity relies on the exposure amount *net* of collateral and any other recoverable amounts given a credit default event. The credit default annual loss distribution is obtained by compounding the default event frequency stochastic process with the loss severity stochastic process.

¹³ Non funded exposures, such as unutilised credit limits or guarantees, can be converted into 'equivalent funded' exposures using the credit conversion factor values proposed in the Standardised Approach.

¹⁴ Segmentation example: corporates, banks, sovereigns, retail, where retail exposures can be further broken down by product (mortgage loans, personal loans, revolving credit, and others).

¹⁵ Collateral types include lien funds, real estate, bank or government guarantees received, and pledged financial securities. Subject to data availability, other recoverable amounts (e.g. loan future repayment estimates) can be considered in addition to collateral.

¹⁶ External credit ratings provided by recognised agencies, or internal ratings, correspond to an annual default probability estimated from historical rating migration long run averages. For sovereigns, where historical defaults are scarce, default probabilities can also be determined from traded credit default swap spreads, see for example those published by Deutsche Bank (www.dbresearch.com), or bond market prices (Vrugt(2011)).

A.2 Distribution compounding and Panjer Recursive Method: Theoretical framework

A.2.1 Distribution compounding

First we lay down the Actuarial Model notation. Random variable N represents the number of credit default events within a risk horizon of one year, $\{X_1, X_2, \dots, X_N\}$ represent random individual losses given default, being independent and identically distributed, and S denotes the annual loss amount aggregated from all obligors in a loan portfolio i.e. $S = X_1 + X_2 + \dots + X_N$. The objective is to establish the distributional form of S .

Denote the probability generating function (pgf) of N by $G_N(z)$. Similarly, denote the pgf of $\{X_1, X_2, \dots, X_N\}$ by $G_X(z)$. By assuming that $\{X_1, X_2, \dots, X_N\}$ do not depend on N , the random sum S has a pgf $G_S(z)$ where

$$G_S(z) = G_N(G_X(z)). \quad (1)$$

The term 'compounding' essentially means that the pgf of the distribution of S , $G_S(z)$, results by the process of compounding two discrete distributions namely $G_N(z)$, called the primary distribution, and $G_X(z)$ called the secondary distribution. Compounding therefore creates the annual loss distribution driven by the default annual frequency process N and the individual credit loss (given default) process X .

Note that the pgf of a discrete random variable S is defined as

$$G_S(z) := \mathbf{E}(z^S) = \sum_{s=0}^{\infty} P_s z^s \quad (2)$$

where P_s is the probability mass function (pmf) of S . One key property is that the pmf of S is recovered by taking the s -th derivative of G_S , that is

$$P_s := \text{Prob}(S = x) = \frac{1}{s!} \left[\frac{d^s G_S(z)}{dz^s} \right]_{z=0} \quad (3)$$

From the above property it follows that if random variables S_1 and S_2 have identical pgfs, i.e. $G_{S_1}(z) = G_{S_2}(z)$, they must also have identical distributions i.e. $P_{S_1} = P_{S_2}$. Thus if the pgf of S can be determined, the pmf of S is also deterministic which is the ultimate objective.

A.2.2 Panjer Recursive Method

This method compounds the $(\alpha, b, 0)$ class of discrete (frequency) distributions with any distributional form on individual losses $\{X_1, X_2, \dots, X_N\}$ as long as the latter are independent and identically distributed and also independent from their occurrence frequency N . The $(\alpha, b, 0)$ class consists of all counting random variables whose pmf fulfills:

$$\text{Prob}(N = n) = \left(a + \frac{b}{n} \right) \text{Prob}(N = n - 1), \quad \text{for } n \geq 1 \quad (4)$$

for some real valued constants a and b such that $a + b \geq 0$ with the initial value $\text{Prob}(N = 0)$ being determined such that $\sum_{n=0}^{\infty} \text{Prob}(N = n) = 1$.

The counting stochastic processes that satisfy (4) are:

- i. Poisson distribution;
- ii. Binomial distribution;
- iii. Negative Binomial distribution.

Sundt and Jewell (1981) presented a re-parameterisation of these three distributions with reference to the $(\alpha, b, 0)$ class of frequency distributions:

- i. Poisson distribution with parameter λ is reparameterised to: $a = 0$; $b = \lambda$
- ii. Binomial distribution with parameters (N, p) is reparameterised to: $a = -p/(1-p)$; $b = (N+1)p/(1-p)$
- iii. Negative Binomial distribution with parameters (r, p) is reparameterised to: $a = p$; $b = (r-1)p$.

Panjer (1981) Theorem:

Consider the compound distribution with probability density function (pdf)

$$f_S(x) = \sum_{n=1}^{\infty} \text{Prob}(N = n) f_x^{*n}(x) \quad (5)$$

for an arbitrary individual loss severity distribution with pdf f_x defined on support $x > 0$. The n -fold convolution of f_x , meaning the probability that the sum of n random variables independent and identically distributed with pdf f_x , that will take on value x is denoted by $f_x^{*n}(x)$. Given (4) and (5) the following recursion holds with regards to the pdf of aggregate loss S :

$$f_S(x) = \text{Prob}(N = 1) f_x(x) + \int_0^x \left(a + \frac{by}{x}\right) f_x(y) f_S(x - y) dy, \text{ for } x > 0. \quad (6)$$

More importantly, with regards to practical applications of this theorem, if the loss severity distribution is discrete, i.e. 'arithmetic', and defined on the positive integers, i.e. $x \in \mathbb{N}_1$, the corresponding recursive definition of the pmf of aggregate loss S reduces to:

$$f_S(x) = \sum_{y=1}^x \left(a + \frac{by}{x}\right) f_x(y) f_S(x - y), \text{ for } x = 1, 2, 3, \dots \quad (7)$$

with $f_S(0) = \text{Prob}(N = 0)$. The original algebraic proof for this theorem can be found in Panjer (1981).

To discretize the loss severity distribution, loss values must be mapped onto a lattice with lattice-width $L \in \mathbb{N}_1 = \{1, 2, 3, \dots\}$. In this case f_x , which is defined on $\{X_1, X_2, \dots, X_N\}$, can be written as $f_x = \text{Prob}(X = nL = x)$ for $x > 0$ and the number of computations required to obtain $f_S(x)$ is of order x . Whereas the number of computations required to obtain $f_S(x)$ is of order x^2 for the original Panjer form specified by (6). For large values of x , the reduction in computations is dramatic which explains the strong preference to a discrete loss severity distribution.

A.2.3 Panjer Recursive Method result application

The $(\alpha, b, 0)$ class of frequency distributions could be used for modelling the credit default event counts N . Two cases are of particular interest in credit risk modelling:

Case 1: Poisson distribution, suitable when annual default probabilities in a loan portfolio are stationary and can be approximated by long run averages of historical default counts.

Case 2: Negative Binomial distribution, suitable when annual default probabilities vary in time.

The following recursive forms for the pmf of aggregate loss S are obtained by applying the Panjer theorem to the re-parameterised Poisson and Negative Binomial with reference to the $(\alpha, b, 0)$ class of frequency distributions:

Case 1, Poisson, $a=0; b=\lambda$

$$\begin{aligned} f_S(x) &= \sum_{y=1}^x \left(a + \frac{by}{x} \right) f_X(y) f_S(x-y), \quad x = 1, 2, 3, \dots \\ &= \frac{\lambda}{x} \sum_{y=1}^x y f_X(y) f_S(x-y) \end{aligned} \quad (8)$$

where $f_S(x=0) = \text{Prob}(N=0) = e^{-\lambda}$.

Case 2, Negative Binomial, $a=p; b=(r-1)p$

$$\begin{aligned} f_S(x) &= \sum_{y=1}^x \left(a + \frac{by}{x} \right) f_X(y) f_S(x-y), \quad x = 1, 2, 3, \dots \\ &= \sum_{y=1}^x \left(p + \frac{(r-1)py}{x} \right) f_X(y) f_S(x-y) \\ &= p \sum_{y=1}^x \left(1 + \frac{(r-1)y}{x} \right) f_X(y) f_S(x-y) \end{aligned} \quad (9)$$

where $f_S(x=0) = P(N=0) = (1-p)^r$.

A.3 Model implementation

The Panjer Recursive Method results applied to the Poisson (case 1) and Negative Binomial (case 2), as described in Section A.2.3, are now put into the context of the Actuarial Model.

A.3.1 Default events with fixed default probabilities & resulting credit default annual loss distribution (Case 1: Poisson credit default event frequency)

The pmf for the credit default annual losses, when default occurrences are assumed to follow a Poisson distribution, is specified by

$$\text{Prob}(\text{aggr. loss} = nL) := P_n = \sum_{j: v_j \leq n} \frac{v_j \lambda_j}{n} P_{n-v_j}, \quad \text{for } n \in \mathbb{N}, \quad (10)$$

where the zero loss probability $P_0 = \exp(-\sum_{j=1}^m \lambda_j)$.

The recursive result P_n specified by (10) is obtained directly from the Panjer Theorem applied on the Poisson frequency distribution. The functional form of P_n can be specified by $f_S(x)$ in (8) since

- each exposure band j is characterised by its own λ_j and v_j ;

- the probability for an obligor default loss to take on the value v_j within band j is 100%, i.e. $f_X(y) = 1$ in (8), as the individual loss severity distribution is approximated by a single value namely the band exposure amount v_j .

The Actuarial model can be implemented in the following ten steps:

Step 1: Divide each and every obligor c exposure amount X_c by a unit of exposure L , e.g. set $L=100,000$ Euro. Denote the scaled exposure amounts by $v_c' = X_c / L$.

Step 2: Given the value of the annual default probability average p_c calculate the expected loss ε_c by $\varepsilon_c = v_c' p_c$ for each obligor c .

Step 3: Round up the scaled exposure amounts v_c' to the nearest integer and denote these by v_c .

Step 4: Create exposure bands $\{1L, 2L, 3L, \dots, (n-1)L, nL\}$ indexed respectively by $j=1,2,3,\dots,m-1,m$ so that $m=n$.

Step 5: Consider all obligors sharing the same v_c value and assign them into a common exposure band v_j . Each band j therefore corresponds to a rounded common exposure v_j i.e. $v_j \equiv v_c$. The number of obligors with common exposure v_j are identified.

Step 6: Calculate the *total* expected loss ε_j for band j by adding up the expected loss values ε_c for all obligors c belonging in band j , i.e. is $\varepsilon_j = \sum_{c:v_c=v_j} \varepsilon_c$

Step 7: The average number of defaults in band j , denoted by λ_j , is approximated by the ratio between the total expected loss ε_j and the common exposure band amount v_j i.e.

$$\lambda_j = \frac{\varepsilon_j}{v_j} = \frac{\sum_{c:v_c=v_j} \varepsilon_c}{v_j} = \sum_{c:v_c=v_j} \frac{\varepsilon_c}{v_c} \quad (11)$$

so that the total number of default occurrences in the entire portfolio over the yearly risk horizon is $\lambda = \sum_{j=1}^m \lambda_j$.

Step 8: Each band j is treated separately as a sub-portfolio of credit exposures. Based on the calculated v_j and λ_j , the probability P_n to observe a total loss nL over a risk horizon of one year must be evaluated. The number of unit exposure L multiples is upper bounded by $n = \text{card}\{\frac{\sum_c X_c}{L}\}$. The evaluation of probability P_n relies on the pgf for band j , which is basically a function of the known parameters v_j and λ_j .

Step 9: Derive the distribution of the annual credit default losses for the entire loan portfolio by multiplying the pgfs for each exposure band $j=1,2,3,\dots,m$ since the exposure bands j are independent.

Step 10: The pmf for the annual credit default loss distribution, P_n , is calculated recursively using (10).

A.3.2 Default events with variable default probabilities (Case 2: negative Binomial credit default event frequency)

Systematic risk factors impact the financial standing and credit quality of obligors in a credit portfolio to a different extent. It is therefore necessary to quantify the extent to which an obligor in the portfolio is impacted by such systematic risk factors which typically relate to the state of the economy in a particular industry sector or in a particular country. For example, the state of the economy in the real estate and construction industry sector is more likely to influence uniformly the financial standing and credit quality of obligors operating in that sector rather than obligors operating in the hotel industry, telecommunications or other unrelated industry sectors.

The extent of the impact of systematic risk factors on obligor credit quality is quantified through the use of default probability standard deviations¹⁷. With reference to the example above, a possible deterioration in the real estate and construction sector would mean a higher default probability standard deviation for obligors operating in that sector compared to the default probability standard deviation of other obligors in the portfolio operating in different sectors of the economy.

Obligor exposures in a portfolio are assigned to different (non-overlapping) sectors, and for each sector the following model assumptions are made:

- Obligor exposures coming from the same sector are assumed to have only one risk factor in common – e.g. same industry sector or same country of operations.
- That single systematic risk factor explains the variability of the average annual default probability in that sector, which is treated as being stochastic with mean μ_k and standard deviation σ_k .
- The systematic risk factor impacts μ_k and σ_k in sector k .
- The standard deviation σ_k quantifies the extent to which the obligor actual annual default probabilities in sector k are anticipated to be higher than their corresponding long run annual average. The higher the σ_k value the higher the spread of the actual annual default probability rate from its long run annual average value, indicating a troubled sector in which the credit quality of obligors has deteriorated or is expected to deteriorate over the next year.

The methodology is implemented in a similar fashion as in case 1 under the Poisson default count assumption. Except that in case 2, under the Negative Binomial default count assumption, the concept of sectors is added. Essentially the model implementation steps, listed in Section A.3.1, are now performed for the obligors in each sector separately, and the results are then aggregated across the k sectors. From a technical perspective, the Negative Binomial distributional form is

¹⁷ In contrast to most credit risk models that rely on correlations for capturing systematic risks, see for example Vasicek (1989), JP Morgan (1997), Dwyer et al (2004), Thompson et al (2001a).

the result of a Poisson distribution whose parameter (count average) is not constant but stochastic, one that follows a Gamma distribution.

The average annual default frequency for the entire loan portfolio is denoted by λ as in the previous notation except that now λ represents a random variable. The random variables $\{\lambda_k\}_{k \geq 1}$ represent the average annual default frequency in each sector k for sectors $k=1,2,3,\dots,K$. The distribution of λ_k is specified by the mean μ_k and standard deviation σ_k , where μ_k represents the long run annual average default frequency over several years. In fact the parametric distribution chosen for λ_k is the Gamma (r_k, s_k) with parameters r_k and s_k . The notation for exposure band $v_j^{(k)}$, total expected loss $\varepsilon_j^{(k)}$, band indicator $j=1,\dots,m^{(k)}$, remain the same as before only that now superscript k indicates the sector for exposure band j . The estimation of parameters μ_k and σ_k from known obligor data, and the estimation of r_k and s_k from the already estimated μ_k and σ_k are described next.

Estimating μ_k and σ_k :

The long run average number of defaults μ_k in sector k is given by an expression that is similar to that of λ_j , specified by [11], that is

$$\mu_k = \sum_{j=1}^{m^{(k)}} \frac{\varepsilon_j^{(k)}}{v_j^{(k)}} = \sum_c \frac{\varepsilon_c}{v_c} \quad (12)$$

this way adding up the long-run average default probabilities for all the obligor exposures in sector k .

Next, the standard deviation σ_k of the total default rate in sector k is estimated. What is actually known from obligor data is the annual default probability standard deviation σ_c for each and every obligor in the loan portfolio. To estimate σ_k from σ_c the random variable λ_c , denoting the average default probability for obligor c , can be expressed as a proportion to λ_k since the latter is the only source of randomness explaining the variability of obligor default probabilities within sector k . Thus,

$$\begin{aligned} \lambda_c \propto \lambda_k &\Rightarrow \lambda_c = Q \lambda_k && \text{where, constant } Q = \frac{\varepsilon_c}{v_c} \frac{1}{\mu_k} \\ \Rightarrow \text{Var}(\lambda_c) &= \left(\frac{\varepsilon_c}{v_c}\right)^2 \frac{1}{\mu_k^2} \text{Var}(\lambda_k) \\ &= \left(\frac{\varepsilon_c}{v_c \mu_k}\right)^2 \text{Var}(\lambda_k) \\ \Rightarrow \sigma_c &= \sqrt{\text{Var}(\lambda_c)} = \left(\frac{\varepsilon_c}{v_c \mu_k}\right) \sigma_k \end{aligned} \quad (13)$$

Taking the sum over all obligors in sector k yields

$$\sum_c \sigma_c = \sum_c \left(\frac{\varepsilon_c}{v_c \mu_k}\right) \sigma_k = \sigma_k \frac{1}{\mu_k} \sum_c \left(\frac{\varepsilon_c}{v_c}\right) = \sigma_k. \quad (14)$$

which means that the standard deviation of the average annual default rate in a sector k can be estimated by the sum of σ_c of all obligors c belonging in sector k .

Estimating r_k and q_k :

The random variable λ_k is distributed by Gamma (r_k, q_k) where parameters $r_k > 0$ and $0 < q_k < 1$. The Gamma distribution mean and variance are denoted by μ_k and σ_k respectively, and their functional form is specified below:

$$\mu_k = r_k q_k \Rightarrow r_k = \frac{\mu_k}{q_k}$$

$$\sigma_k^2 = r_k q_k^2.$$

From the above definitions it follows that for sector k the Gamma distribution parameters r_k and q_k can be estimated from the now known μ_k and σ_k :

$$r_k = \frac{\mu_k^2}{\sigma_k^2}; \quad q_k = \frac{\sigma_k^2}{\mu_k}. \quad (15)$$

The pgf for the frequency of default occurrences from obligors in the k^{th} sector is

$$G_N(z)^{(k)} = \left(\frac{1 - p_k}{1 - p_k z} \right)^{r_k} \quad (16)$$

where $p_k = \frac{q_k}{1 + q_k}$ and the pmf underlying the above pgf is

$$\text{Prob}(N = n) = (1 - p_k)^{r_k} \binom{n + r_k - 1}{n} p_k^n \quad (17)$$

where both (16) and (17) represent the Negative Binomial distribution (with parameters r_k, p_k) introduced in Section A.2.3¹⁸.

In the context of the notation in this section, a Negative Binomial random variable belongs in the $(\alpha, b, 0)$ class with $\alpha = p_k$ and $b = (r_k - 1) p_k$. The theoretical proof of the derivation of the Negative Binomial distribution from the Poisson-Gamma mixture appears in Klugman et al (2012) and references therein.

Finally the pgf for default events that may arise from the entire portfolio which comprises of K sectors is written as

$$G_N(z) = \prod_{k=1}^K G_N(z)^{(k)} = \prod_{k=1}^K \left(\frac{1 - p_k}{1 - p_k z} \right)^{r_k} \quad \text{where } p_k = \frac{q_k}{1 + q_k} \quad (18)$$

The derivation of (18) relies on (16) and the assumption of conditional independence; that is the default event occurrences within a sector are independent from those in the other sectors conditional on the distribution of the

¹⁸ The Binomial coefficient in [12] is $\binom{n + r_k - 1}{n} = \frac{(n + r_k - 1)!}{n! (r_k - 1)!} = \frac{(n + r_k - 1)(n + r_k - 2) \cdots r_k}{n!}$

average annual default frequency $\{\lambda_k\}_{k \geq 1}$ in each sector k being specified by Gamma (r_k, q_k) with mean μ_k and standard deviation σ_k .

A.3.3 Resulting credit default annual loss distribution (Case 2: negative Binomial credit default event frequency)

Recall from equation (1) that the pgf of the credit default annual loss distribution of S , namely $G_S(z)$, results by the process of compounding G_N (being the primary pgf of annual frequency process N) and G_X (being the secondary pgf of individual credit default event loss process X). Equation (1) is re-written at the level of sector k as follows

$$G_S(z)^{(k)} = G_N^{(k)}(G_X(z)^{(k)}) \quad (19)$$

where

$$G_X(z)^{(k)} = \frac{\sum_{j=1}^{m^{(k)}} \left(\frac{\mathcal{E}_j^{(k)}}{\nu_j^{(k)}} \right) z^{\nu_j^{(k)}}}{\sum_{j=1}^{m^{(k)}} \left(\frac{\mathcal{E}_j^{(k)}}{\nu_j^{(k)}} \right)} = \frac{\sum_{j=1}^{m^{(k)}} \left(\frac{\mathcal{E}_j^{(k)}}{\nu_j^{(k)}} \right) z^{\nu_j^{(k)}}}{\mu_k} \quad (20)$$

with the right-most part of (20) resulting from (12) in Section A.3.2.

At the same time, the independence of sectors $k=1,2,\dots,K$ allows expressing the pgf of the entire loan portfolio as the product of the sectoral pgfs, i.e.

$$G_S(z) = \prod_{k=1}^K G_S(z)^{(k)} = \prod_{k=1}^K G_N^{(k)}(G_X(z)^{(k)}) \quad (21)$$

where the right-most part of (21) results from (19).

By substituting (20) and (16) into (21) the pgf of the aggregate credit default annual loss S on the entire portfolio is specified as follows

$$G_S(z) = \prod_{k=1}^K \left(\frac{1-p_k}{1-p_k G_X(z)^{(k)}} \right)^{r_k} \\ = \prod_{k=1}^K \left(\frac{1-p_k}{1-p_k \frac{\sum_{j=1}^{m^{(k)}} \left(\frac{\mathcal{E}_j^{(k)}}{\nu_j^{(k)}} \right) z^{\nu_j^{(k)}}}{\mu_k}} \right)^{r_k} \quad (22)$$

The pmf of the credit default annual loss for sector k , namely $Prob(aggr. loss in sector k = nL) := P_n^{(k)}$, can be expressed in a recursive form by applying the Panjer Theorem to the Negative Binomial frequency distribution. The functional form of $P_n^{(k)}$ can be specified using $f_s(x)$ in (9) since

- each exposure band j in sector k is characterised by its own parameters r_k , p_k and $v_j^{(k)}$;
- a Negative Binomial random variable belongs in the $(\alpha, b, 0)$ class with $\alpha=p_k$ and $b=(r_k - 1) p_k$ for a credit portfolio with $k=1, \dots, K$ sectors;
- the probability for an obligor default loss to take on the value $v_j^{(k)}$ within band j in sector k is 100%, i.e. $f_X(y) = 1$ in (9), as the individual loss severity distribution is approximated by a single value namely the band exposure amount $v_j^{(k)}$.

Based on the above observations, equation (9) is re-arranged as follows

$$\begin{aligned} Prob(aggr. loss in sector k = nL) &:= P_n^{(k)} \\ &= p_k \sum_{j:v_j \leq n} \left(1 + \frac{(r_k - 1)v_j^{(k)}}{n} \right) P_{n-v_j^{(k)}}^{(k)} \\ &= \frac{p_k}{n} \sum_{j:v_j \leq n} (n + (r_k - 1)v_j^{(k)}) P_{n-v_j^{(k)}}^{(k)} \end{aligned} \quad (23)$$

with initial value set at $P_0 = (1 - p_k)^{r_k}$. The above pmf is numerically stable because $n - v_j^{(k)} \in \mathbb{N}_0$.

The parameters r_k, p_k in pgf (22) and pmf (23) are estimated by μ_k and σ_k using (15). The parameters μ_k, σ_k are in turn estimated by the input data using (12) and (14) respectively.

A.4 Actuarial Model extension - using Extreme Value Theory (EVT) to infer the far end tail area of the annual loss distribution

The Panjer recursion specified theoretically in (7), and in the context of the Actuarial Model in (10) and (23), employ an arithmetic *loss severity* distribution as discussed in Section A.2.2. However, arithmetic loss severity distributions are of ‘phase type’, meaning in practice that they tend to be uniformly dense with light tails, hence not always suitable for approximating loss distributions with heavy tails i.e. extreme loss values. In technical terms, their Laplace transform exists in a neighborhood of 0 as explained in Embrechts and Frei (2009), who argue that phase type distributions, including those used in the Panjer recursion, are more suitable for modeling small-sized insurance claims¹⁹ or other risk losses.

As a result the quantiles of the annual loss distribution produced by the Actuarial Model presented in this paper cannot look at the far end part of the tail. Quantile loss estimates up to the 99.9% confidence level (i.e. 99.9% one year VaR) are

¹⁹ For a review of the use of phase type distributions in insurance risk theory see Bladt (2005).

usually reliable, but extrapolating beyond that level can often cause numerical instability issues to the Panjer recursion formula. Next we propose the use of EVT for inferring the far end tail of the annual loss distribution beyond 99.9%.

A.4.1 EVT essentials

EVT is beyond the scope of this paper, nevertheless we introduce only the information necessary that would allow the formulation of a tail sensitive VaR.

The most frequently used extreme value distribution among EVT is the Generalised Pareto Distribution or GPD. The GPD can approximate the distribution of credit default annual losses exceeding in value a high threshold u . Thus, if X denotes a random credit default annual loss, the corresponding annual loss excess $X-u:=Y$, conditional on $X \geq u$, is assumed to follow a GPD with shape and scale parameters ξ and $\beta > 0$ respectively. The cumulative distribution function (cdf) $G_{\xi,\beta}(y)$ for an observed excess y has the following form

$$P(Y \leq y | (u, \beta, \xi)) := G_{\xi,\beta}(y) = \begin{cases} 1 - \left(1 + \xi \frac{y}{\beta}\right)^{-1/\xi} & \text{if } \xi \neq 0 \\ 1 - \exp\left(-\frac{y}{\beta}\right) & \text{if } \xi = 0 \end{cases}$$

$$\text{where } y \in \begin{cases} [0, \infty] & \text{if } \xi \geq 0 \\ [0, -\frac{\beta}{\xi}] & \text{if } \xi < 0. \end{cases} \quad (24)$$

A.4.2 Constructing a tail sensitive quantile risk measure

The cdf $G_{\xi,\beta}(y)$ specified in (24) approximates the empirical cdf denoted by $F_u(y)$ of the tail of the annual loss distribution produced from the Actuarial Model. The tail area comprises of large annual loss values that satisfy $X \geq u$. To obtain a closed form expression for the quantile of the *underlying* distribution of annual losses, we need first to express the corresponding cdf, here denoted by $F(u+y)$, in terms of $F_u(y)$. So we use the following relationship which follows directly from Bayes' rule

$$P(X - u \leq y | x > u) := F_u(y) = \frac{F(u+y) - F(u)}{1 - F(u)}, \quad \text{for } 0 \leq y < \infty \quad (25)$$

The equation above is solved with respect to $F(u+y)$. For mathematical convenience this is re-written in terms of survivor functions $\bar{F}() := 1 - F()$ as $\bar{F}(u+y) = \bar{F}(u) \bar{F}_u(y)$. The survivor function $\bar{F}(u+y)$ is approximated semi-parametrically. The tail distribution, conditional on $X \geq u$, is approximated parametrically by the GPD, i.e. $\bar{F}_u(y) \approx 1 - G_{\xi,\beta}$. The unconditional distribution is approximated empirically by the proportion of the annual losses exceeding

threshold u , i.e. $\bar{F}(u) \approx n_u/n$ where n_u denotes the number of annual losses whose value is greater than u and n the number of all losses making up the credit risk annual loss distribution. Therefore the survivor function $\bar{F}(u+y)$ of the underlying distribution is approximated by

$$\bar{F}(u+y) = \bar{F}(u) \bar{F}_u(y) \approx \frac{n_u}{n} \left(1 + \xi \frac{y}{\beta}\right)^{-1/\xi} \quad (26)$$

By inverting this expression and solving for loss x we obtain the quantile $x_q := u + y_q$, for $y_q \geq 0$, that corresponds to confidence level q (e.g. $q=99.98\%$). So by solving for $x_q = \bar{F}^{-1}(1-q)$ in

$$\bar{F}(u+y_q) := \bar{F}(x_q) \approx \frac{n_u}{n} \left(1 + \xi \frac{x_q - u}{\beta}\right)^{-1/\xi} = 1 - q \quad (27)$$

we obtain, for $\xi \geq 0$, the following Extreme VaR (EVAR) expression at confidence level q

$$EVaR_q := x_q = u + \frac{\beta}{\xi} \left[\left(\frac{n}{n_u} (1-q) \right)^{-\xi} - 1 \right] \quad (28)$$

The shape ξ and scale β GPD parameters in (28) are replaced by their sample estimates computed using numerical procedures. For a detailed exposition on EVT and the estimation procedures of extreme value distributions see Kyriacou and Medova (2000, 2002) and Kyriacou (2001).

Appendix B: Vasicek One Factor Model

B.1 Vasicek One Factor Model overview

This model uses the same data employed by the Actuarial Model described in Appendix A only that now for each obligor in the credit portfolio

- (a) exposures are gross of collateral value;
- (b) collateral is accounted for separately through the Loss Given Default (LGD) parameter defined as the ratio of obligor unsecured exposure to the exposure amount;
- (c) credit default correlation replaces the default probability standard deviation.

B.2 Vasicek One Factor Model implementation

This model relies on the Merton (1974) approach which is founded on the idea that credit defaults are generated by obligor asset returns when the value of total assets of an obligor falls below the total liabilities.

B.2.1 Model specification

The Vasicek One Factor model assumes the existence of one common factor among obligors in a portfolio, namely the state of the economy. Given a credit portfolio with N obligors (and N exposures) the i^{th} obligor normalized asset returns Z_i are specified by

$$\text{where } Z_i = \sqrt{\rho} C + \sqrt{1 - \rho} \varepsilon_i \quad (29)$$

- C is a random variable representing the state of the economy being the systematic risk factor affecting all N obligors in the portfolio;
- ε_i is also stochastic representing i^{th} obligor-specific idiosyncratic risk factors such as management effectiveness, business mix, strategy, and innovation;
- Random variables $\{C, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ are taken to be independent and identically $N(0,1)$ distributed;
- ρ is the 'loading' parameter controlling asset return correlation between obligors in the portfolio. In the Vasicek One Factor Model this parameter ranges from 0 to 1 and controls the extent to which asset returns are explained by the systematic risk factor (state of the economy, namely) versus the idiosyncratic risk factors. In practice, ρ is usually set within the range [10%-25%], meaning that 10%-25% of a change in i^{th} obligor asset returns Z_i are explained by changes in the state of the economy.

The correlation structure of credit default probabilities for N obligors in the portfolio is determined by the correlation structure of their respective asset returns. By conditioning asset returns on a single systematic risk factor being the state of the economy, $\{(Z_1, Z_2, \dots, Z_n) | C=c\}$ become conditionally independent given a bad state c of the economy C . This means future default events $\{Z_1 < V_{1\text{default}}, Z_2 < V_{2\text{default}}, \dots, Z_N < V_{N\text{default}}\}$ may occur independently conditional on $C=c$. Thus, individual obligor default probabilities $\{p_1, p_2, \dots, p_N\}$ are independent conditional on $C=c$.

The i^{th} obligor conditional default probability can therefore be expressed as

$$\begin{aligned} P(Z_i < V_{i\text{default}} | C = c) &= P\left(\left(\sqrt{\rho}C + \sqrt{1 - \rho}\varepsilon_i\right) < V_{i\text{default}} \mid C = c\right) \\ &= P\left(\left(\sqrt{\rho}C + \sqrt{1 - \rho}\varepsilon_i\right) < \Phi^{-1}(p_i) \mid C = c\right) \end{aligned} \quad (30)$$

Re-arranging the contents in the parantheses

$$\begin{aligned} \sqrt{\rho}c + \sqrt{1 - \rho}\varepsilon_i &< \Phi^{-1}(p_i) \\ \Rightarrow \varepsilon_i &< \frac{\Phi^{-1}(p_i) - \sqrt{\rho}c}{\sqrt{1 - \rho}} \end{aligned} \quad (31)$$

where $\varepsilon_i \sim N(0,1)$ as stated earlier. From (30) and (31) it follows that the i^{th} obligor conditional default probability can be written as

$$P(Z_i < V_{i, default} | C = c) = P\left(\varepsilon_i < \frac{\Phi^{-1}(p_i) - \sqrt{\rho}c}{\sqrt{1-\rho}}\right) \equiv \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho}c}{\sqrt{1-\rho}}\right) \quad (32)$$

Earlier it was mentioned that c represents a bad state of the economy where the (normalised) growth rate of the economy is treated as a Standard Normal random variable i.e. $C \sim N(0,1)$. One could set $-c = \Phi^{-1}(x)$ where x represents the VaR confidence level, i.e. how confident the Bank is that the capital (as measured by VaR) will be sufficient to cover the credit risk losses over the next year. If for example $x = 99.9\%$ it means that $c = -3.1$.

Substituting $-c = \Phi^{-1}(x)$ in (32) yields

$$P(Z_i < V_{i, default} | C = c) = \Phi\left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}}\right) \quad (33)$$

But,

$$\begin{aligned} P(\text{Credit Loss}_i \text{ to Exposure}_i \leq \text{VaR}\%_{i,x}) &= x \\ \Rightarrow \text{VaR}\%_{i,x} &= \Phi\left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}}\right) \end{aligned} \quad (34)$$

where VaR % in (34) is expressed as a percentage of gross (before accounting for collateral) exposure.

Transforming VaR from exposure percentage to monetary units:

In order to compute VaR in monetary terms one should multiply the VaR% specified in (34) by VaR and LGD, i.e.

$$\text{VaR}_{i,x} = \text{EaD}_i \text{ LGD}_i \Phi\left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}}\right) \quad (35)$$

where

- EaD_i is i^{th} obligor's exposure amount at default;
- LGD_i is i^{th} obligor's loss given default rate.

Credit Risk Capital Allocation to Unexpected Losses Only:

Expected Loss (EL) is anticipated to arise under the normal course of business. As such it is attributed as a cost into a risk-adjusted return on capital calculation and essentially represents bad debt provisions. Thus capital is only required for the Unexpected Loss (UL). The EL_i is therefore subtracted from $\text{VaR}_{i,x}$ in (35) where $\text{EL}_i = \text{EaD}_i \text{ LGD}_i p_i$. So,

$$\begin{aligned}
\text{Credit UL Capital}_{i,x} &= \text{VaR}_{i,x} - \text{EL}_i & 69 \\
&= \text{EaD}_i \text{ LGD}_i \Phi \left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}} \right) - \text{EaD}_i \text{ LGD}_i p_i \\
&= \text{EaD}_i \text{ LGD}_i \left(\Phi \left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}} \right) - p_i \right) & (36)
\end{aligned}$$

Equation (36) represents the maximum one year Unexpected Loss with confidence level x from i^{th} obligor possible credit default. The total one year unexpected losses from all possible credit default(s) for a portfolio of N obligors is therefore:

$$\text{Credit UL Capital}_x = \sum_{i=1}^N \text{EaD}_i \text{ LGD}_i \left(\Phi \left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}} \right) - p_i \right) \quad (37)$$

B.2.2 Adjustments for maturity and correlations

The one year unexpected loss specified in (36) for i^{th} obligor and for the entire portfolio in (37)

- (a) make no reference to credit facility maturity, and should be scaled up for longer than-a-year credit risk positions;
- (b) do not specify the estimation of the loading parameter ρ explaining obligor asset return correlations due to common systematic risk factor C .

The issues (a) and (b) listed above are addressed using some key assumptions made by the regulators in the IRB approach.

Maturity Adjustment:

According to the IRB approach, the i^{th} obligor-specific Maturity Adjustment (MAD) factor is defined by

$$\text{MAD}_i = \frac{1 + (m_i - 2.5) * b_i}{1 - 1.5 * b_i}; \text{ where } b_i = (0.11852 - 0.05478 * \ln(p_i))^2 \quad (38)$$

where:

- MAD rate of increase is higher for lower default probabilities p since there is more down-side potential for 'good' obligors over the longer term. MAD takes values in the range [1,1.9] but when $p \geq 5\%$ $\text{MAD} \in [1,1.3]$.
- m represents the remaining maturity in years with values preset in the range [1,5].

Multiplying the MAD factor specified in (38) with (36) and then aggregating this product across all obligors leads to a closed form expression for calculating the adjusted total one year unexpected loss for an N -obligor portfolio,

$$\text{Credit UL Capital}_x = \sum_{i=1}^N \text{EaD}_i \text{ LGD}_i \left(\Phi \left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}} \right) - p_i \right) \left(\frac{1 + (m_i - 2.5) b_i}{1 - 1.5 b_i} \right) \quad (39)$$

Correlation Estimation:

The loading parameter ρ that determines asset return correlations for corporate and sovereign obligors due to the presence of a common systematic risk factor C is estimated by:

$$\text{Correlation, } \rho = 0.12 * \left(\frac{1 - e^{-50 * p_i}}{1 - e^{-50}} \right) + 0.24 * \left(1 - \left(\frac{1 - e^{-50 * p_i}}{1 - e^{-50}} \right) \right) \quad (40)$$

where $\rho \equiv \rho_i$ is taken to be i^{th} obligor-specific through the default probability p_i by design of the IRB.

The principles underlying the estimation of correlation (or loading parameter) ρ are:

- ρ is decreasing for higher default probabilities p since the explanatory power of the idiosyncratic risk factors triggering a credit default event is higher in these cases; i.e. default occurrence for 'bad' obligors depends less on the systematic risk factor C (being the state of the economy) and is more driven by obligor-specific risk factors.
- ρ is preset in the range [12%, 24%]; if $p=0$ then $\rho=24\%$, if $p=100\%$ then $\rho=12\%$. Parameter ρ decays in an exponential fashion, i.e. ρ decreases at a fast rate in the range $p \in [0,5\%]$ and a much slower rate for the range $p > 5\%$.
- For retail exposures $\rho \in [3\%,16\%]$ and the exponential decay is implemented in 25 steps instead of 50 steps specified in (40)²⁰.

B.2.3 Computing the whole credit default annual loss distribution

Each obligor's EL (namely, $EL_i = EaD_i LGD_i p_i$) is added back to (39) in order to establish the adjusted VaR_x representing the portfolio expected and unexpected credit default loss at the selected confidence level x (after the IRB adjustments discussed in Section B.2.2). That is,

$$VaR_x = \sum_{i=1}^N \left(EaD_i LGD_i \left(\Phi \left(\frac{\Phi^{-1}(p_i) + \Phi^{-1}(x)\sqrt{\rho}}{\sqrt{1-\rho}} \right) - p_i \right) \left(\frac{1 + (m_i - 2.5) b_i}{1 - 1.5 b_i} \right) + EaD_i LGD_i p_i \right) \quad (41)$$

whose algebraic form can be simplified further.²¹

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²⁰ A different ρ -value applies in IRB for special classes of retail exposures, for instance $\rho = 15\%$ for Residential Mortgages, and $\rho = 4\%$ for Revolving Retail Exposures.

²¹ Vasicek (1991) has proved a different closed form expression on the portfolio loss rate and VaR that applies to very large and homogeneous credit portfolios, where individual obligor default probabilities are approximately the same and portfolios are not dominated by a few disproportionately large loans.

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