UNIVERSAL SERIES IN $\cap_{p>1} \ell^p$

STAMATIS KOUMANDOS, VASSILI NESTORIDIS$^{†,‡}$, YIORGOS–SOKRATIS SMYRLIS§, AND VANGELIS STEFANOPoulos

ABSTRACT. We give an abstract condition yielding universal series defined by sequences $a = \{a_j\}_{j \in \mathbb{N}}$ in $\cap_{p>1} \ell^p$ but not in $\ell^1$. We obtain a unification of some known results related to approximation by translates of specific functions including the Riemann $\xi$–function or by translates of a fundamental solution of a given elliptic operator in $\mathbb{R}^\nu$ with constant coefficients or by translates of an approximate identity as for example by the translates of normal distributions. Another application gives universal trigonometric series in $\mathbb{R}^\nu$ simultaneously with respect to all $\sigma$–finite Borel measures in $\mathbb{R}^\nu$. Stronger results are obtained by using universal Dirichlet series in one or several variables.

1. INTRODUCTION

Universal series date back almost a century and continue to be an interesting area of research per se and also because of their connections with other areas of Mathematics, such as dynamical systems and operator theory. Recently an abstract framework has been developed [NP05, BGENP] which allows a unified approach and leads on the one hand to simplification of the proofs of almost every known result and on the other hand to obtaining new results. This covers the theorems of Seleznev, an extension of Fekete’s theorem, universal Taylor series, universal Dirichlet series, universal Faber series, universal Laurent series, universal trigonometric series in the sense of Menchoff, universal expansions whose terms are homogeneous harmonic polynomials, universal expansions for $C^\infty$–functions, etc.

In accordance with the abstract theory of universal series, let $\mathcal{X}$ be a topological vector space and assume that its topology is induced by a translation invariant metric $\rho$. Let $\{x_j\}_{j \in \mathbb{N}}$ be a fixed sequence of elements of $\mathcal{X}$. A scalar sequence $a = \{a_j\}_{j \in \mathbb{N}}$ (where $\mathbb{N}$ is the set of non–negative integers) belongs to the class $\mathcal{U}$, if the sequence $\{\sum_{j=0}^n a_j x_j\}$
is dense in \( \mathcal{X} \); then the sequence \( a \) defines a universal series \( \sum_{j=0}^{\infty} a_j x_j \). If \( A \) is a vector subspace of the set of all scalar sequences, satisfying certain properties, then we are interested in sequences \( a = \{a_j\}_{j \in \mathbb{N}} \) in \( A \) which define universal series. The existence of such a sequence is guaranteed by a condition which requires a double approximation ([BGENP, NP05]); in fact, this condition is also necessary. This double approximation is verified, in most of the cases considered, by applying well–known approximation theorems, such as Weierstrass’s Theorem, Runge’s Theorem, Mergelyan’s Theorem, Walsh’s Theorem, Browder’s Theorem, etc.

In the present paper we consider a topological vector space \( E \) and we assume that its topology is induced by a translation invariant metric \( \rho \). We also consider a fixed sequence \( \{x_j\}_{j \in \mathbb{N}} \) in \( E \). We assume that the following condition is valid:

**CONDITION D.** For every finite set \( I \subset \mathbb{N} \), there exist distinct indices \( j_n(i), n \in \mathbb{N}, i \in I \), such that \( x_{j_n(i)} \to x_i \), as \( n \to \infty \).

If Condition D is valid, we set \( \mathcal{X} \) to be the set of all finite linear combinations of \( x_0, x_1, \ldots \), endowed with the metric \( \rho \) from \( E \), and we prove that \( \mathcal{X} \neq \emptyset \) and that there exists a sequence \( a = \{a_j\}_{j \in \mathbb{N}} \) with the following property:

The sequence \( a = \{a_j\}_{j \in \mathbb{N}} \) belongs to \( \cap_{p \geq 1} \ell^p(\mathbb{N}) \) and the partial sums \( \{\sum_{j=0}^{n} a_j x_j\}_{n \in \mathbb{N}} \) are dense in \( \mathcal{X} \).

The result is generic ([Kah00, GE99]). We also have algebraic genericity ([Bay05, BGENP]). It is obvious that the sequence \( \{\sum_{j=0}^{n} a_j x_j\}_{n \in \mathbb{N}} \), is dense not only in \( \mathcal{X} \), but also in the closure \( \overline{X} \) of \( \mathcal{X} \) in \( E \) or in the completion \( X \) of \( \langle \mathcal{X}, \rho \rangle \). This is important for the applications because \( \overline{X} \) or \( X \) are often well–known classical spaces and we obtain thus universal series realizing approximations in these spaces. The sequence \( a = \{a_j\}_{j \in \mathbb{N}} \) belongs to \( \cap_{p \geq 1} \ell^p(\mathbb{N}) \) but, as it is shown by an example, it can not in general belong to \( \ell^1(\mathbb{N}) \).

This paper is organized as follows. In Section 2, we present the main abstract theorem. In Section 3, we give some examples where this theorem applies. In most of these examples the \( x_i \)’s are translates of a given continuous function \( \varphi \) in \( \mathbb{R}^v \). Condition D is guaranteed by the continuity of the translation \( a \to \varphi_a \), where \( \varphi_a(t) = \varphi(t - a) \). In Section 4, we unify the results of [Steb], [NS07] and [NS], and present them as corollaries of our abstract theorem. These results are related to universal approximation (i) by translates of the Riemann \( \zeta \)–function [GT05] or more generally, by translates of any function of the form \( \varphi(z) = c/z + g(z) \), where \( c \in \mathbb{C} \setminus \{0\} \) and \( g \) being an entire function [Steb] or (ii) by translates of the fundamental solutions of an elliptic operator in \( \mathbb{R}^v \) with constant coefficients [NS07] or (iii) by translates of an approximate identity, as for example, one consisting of normal distributions [NS]. The spaces \( X \) in which we realize universal approximation in these three cases are explicitly identified and are classical ones. In Section 5 we present an original classical application of our abstract theorem. This relates to universal trigonometric series in \( \mathbb{R}^v \), with respect to all \( \sigma \)–finite Borel measures simultaneously. The first known result in the non–periodic case can be found in [Edg70] and covers the case \( \nu = 1 \) with respect to Lebesgue measure only. For the periodic case we
refer to [GE99, Kah00, KN00, Men45] and references therein. In Section 6, we use universal Dirichlet series in one or several variables in order to improve the results of Section 5. In Section 6, we strengthen the results of Section 5 by using only the frequencies \( \log j \), \( j \geq e^\beta \), \( j \in \mathbb{N} \), for every \( \beta \in \mathbb{R} \). It is worth-mentioning that a condition similar to Condition D first appeared in [Stea], where we had a tree-type splitting of the space \( E \). In fact, \( x_{m(i)} \) in Condition D may be replaced by a finite combination of \( x_0, x_1, \ldots \), so that the spectra are disjoint and the coefficients are uniformly bounded. Then a similar result may be obtained by a slight modification of the proof.

\[ \text{2. AN ABSTRACT THEOREM} \]

We start with some preliminary results concerning the abstract theory of universal series [BGENP, GE99, NP05], which are required for the statement and the proof of our result.

Let \( \mathcal{X} \) be a topological vector space over the field \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), the topology of which is induced by a translation invariant metric \( \rho \). Let \( \{x_j\}_{j \in \mathbb{N}} \) be a fixed sequence of elements of \( \mathcal{X} \). A sequence \( \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{K}^\mathbb{N} \) belongs to the class \( \mathcal{U} \) if the sequence \( \{\sum_{j=0}^n a_j x_j\}_{n \in \mathbb{N}} \) is dense in \( \mathcal{X} \). Obviously \( \mathcal{U} \) is a subclass of the space \( \mathbb{K}^\mathbb{N} \). We endow \( \mathbb{K}^\mathbb{N} \) with its Cartesian topology. If \( \mathcal{U} \neq \emptyset \), then automatically \( \mathcal{U} \) is a dense \( G_d \) subset of \( \mathbb{K}^\mathbb{N} \), and moreover it contains a dense vector subspace of \( \mathbb{K}^\mathbb{N} \) except the zero sequence ([BGENP, GE99, NP05]).

The necessary and sufficient condition for \( \mathcal{U} \) to be non–empty is that, for every \( m \in \mathbb{N} \), the set of linear combinations of \( x_{m+1}, x_{m+2}, \ldots \) is dense in \( \mathcal{X} \) ([GE99]). Every \( a = \{a_j\}_{j \in \mathbb{N}} \) in \( \mathcal{U} \) defines an unrestricted universal series. However, we are interested in restricted universal series, i.e., we require that the sequence \( a = \{a_j\}_{j \in \mathbb{N}} \) belongs to some vector subspace \( A \) of \( \mathbb{K}^\mathbb{N} \), which is a topological vector space and its topology is induced by a translation invariant metric \( d \). We assume that the space \( \langle A, d \rangle \) satisfies the following postulates:

- **P₁**: The metric space \( \langle A, d \rangle \) is complete.
- **P₂**: For every \( m \in \mathbb{N} \) the projection \( \pi_m : A \to A \), defined as \( \pi_m(\{a_j\}_{j \in \mathbb{N}}) = a_m \), is continuous.
- **P₃**: The space \( C_{00} \subset \mathbb{K}^\mathbb{N} \) of sequences which vanish for all but finitely many \( j \in \mathbb{N} \) is a subset of \( A \).
- **P₄**: \( C_{00} = A \).

**Definition 1.** A sequence \( a = \{a_j\}_{j \in \mathbb{N}} \) in \( A \) belongs to the class \( \mathcal{U}_A \) if for every \( x \in \mathcal{X} \), there exists a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) in \( \mathbb{N} \) such that

\[(i)\quad \sum_{j=0}^{\lambda_n} a_j x_j \to x, \text{ as } n \text{ tends to infinity.}\]

\[(ii)\quad \sum_{j=0}^{\lambda_n} a_j e_j \to a, \text{ as } n \text{ tends to infinity, where } e_j = \{\delta_{ji}\}_{i \in \mathbb{N}}.\]

The class \( \mathcal{U}_A \) is the class of restricted universal series.

**Remarks 1.**

- **b.** In Definition 1, it is equivalent to assume that the sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \) is strictly increasing.
- **a.** If the space \( A \) satisfies the postulates \( P₁, P₂, P₃ \) and \( P₄' \), where
Using Condition D, we can find, for every \( p \in \mathbb{N} \), tends to \( a \), as \( n \) tends to infinity, for every \( a \in A \), then \( P_4 \) is also satisfied. In this case \( \mathcal{U}_A = \mathcal{U} \cap A \). Examples of such \( A \) are \( \mathbb{K}^N \), \( c_0 \), \( \ell^q \), \( q \in (1, \infty) \) and \( \cap_{p=1}^{\ell^p} \) endowed with their natural metrics. For an example where \( \mathcal{U}_A \neq \mathcal{U} \cap A \), we refer to [BGENP].

**Theorem 1.** ([BGENP, NP05]) Under the above assumptions and notations, if the class \( \mathcal{U}_A \) is non–empty, then automatically \( \mathcal{U}_A \) is a dense \( G_\delta \) in \( A \) and contains a dense vector subspace except of zero. The condition that follows is necessary and sufficient for \( \mathcal{U}_A \neq \emptyset \): For every \( x \in X \) and every \( \varepsilon > 0 \), there exist \( M \in \mathbb{N} \) and \( \gamma_0, \ldots, \gamma_M \in K \), so that

\[
\begin{align*}
(iii) & \quad \rho(\sum_{j=0}^{M} \gamma_j x_j, x) < \varepsilon \quad \text{and} \\
(iv) & \quad d(\sum_{j=0}^{M} \gamma_j e_j, 0) < \varepsilon.
\end{align*}
\]

In our first abstract result, we establish the existence of universal series in \( \cap_{p=1}^{\ell^p} \), provided that the Condition \( D \) is satisfied and \( \mathcal{U} \neq \emptyset \).

**Proposition 2.** If all the previous assumptions, including Condition \( D \), hold, and also \( \mathcal{U} \) is non–empty, then \( \mathcal{U}_A \cap_{p=1}^{\ell^p} \) is also non–empty. In particular, \( \mathcal{U}_A \cap_{p>1}^{\ell^p} \) is a dense \( G_\delta \) in \( \cap_{p>1}^{\ell^p} \), endowed with its natural topology, and contains a dense vector subspace except of zero. Furthermore, similar statements hold if \( \cap_{p>1}^{\ell^p} \) is replaced by \( \ell^q \), \( q \in (1, \infty) \), \( c_0 \) or \( \mathbb{K}^N \), endowed with their natural topologies.

**Proof.** The spaces \( \ell^q \), \( 1 < q < \infty \), \( c_0 \), \( \mathbb{K}^N \) satisfy the postulates \( P_1 - P_4 \). The space \( \cap_{p=1}^{\ell^p} \) is a Fréchet space and its topology is induced by the translation invariant metric

\[
d(a, b) = \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{\|a - b\|_{1+1/j}}{1 + \|a - b\|_{1+1/j}},
\]

where \( \| \cdot \|_p \) is the norm of \( \ell^p \). Moreover it is easily seen that it satisfies the postulates \( P_1 - P_4 \). Since \( \cap_{p>1}^{\ell^p} \) is contained in all other spaces, it suffices to show that \( \mathcal{U}_A \cap_{p>1}^{\ell^p} \neq \emptyset \). In particular, it suffices to show that conditions \((iii)\) and \((iv)\) of Theorem 1 are valid.

Let \( x \in X \) and \( \varepsilon > 0 \). Since \( \mathcal{U} \neq \emptyset \), it follows that there exists an \( M \in \mathbb{N} \) and \( \gamma_0, \ldots, \gamma_M \in K \), so that \( \rho(\sum_{j=0}^{M} \gamma_j x_j, x) < \varepsilon/2 \). Let \( N \in \mathbb{N} \) which will be specified later. Using Condition \( D \), we can find, for every \( j = 0, \ldots, M \), indices \( j_1, \ldots, j_N \), such that

\[
\rho(\gamma_j x_{j_v} \sum_{v=1}^{N} x_{j_v}) < \frac{\varepsilon}{2(M+1)}.
\]

Then, if follows that \( \rho \left( x, \sum_{j=0}^{M} \sum_{v=1}^{N} \frac{\gamma_j}{N} x_{j_v} \right) < \varepsilon \), which provides condition \((iii)\). We also have

\[
\left\| \sum_{j=0}^{M} \sum_{v=1}^{N} \frac{\gamma_j}{N} e_{j_v} \right\|_p = \left( \sum_{j=0}^{M} \left| \frac{\gamma_j}{N} \right|^{p} \right)^{1/p},
\]

for every \( p > 1 \). Let \( L \) be such that \( \sum_{j=L+1}^{\infty} 2^{-j} < \varepsilon/2 \). The positive integer \( N \) is now chosen so that the right–hand side of (2.1) is less than \( \varepsilon/2 \), for every \( p = 1 + 1/j \), where
with the metric. Due to Proposition 2, it suffices to show that Condition D implies that

\[ d\left(\sum_{j=0}^{M} \sum_{v=1}^{N} \frac{\gamma_j}{N} e_{jv}, 0\right) < \frac{\varepsilon}{2} \sum_{j=1}^{L} \frac{1}{2^j} + \frac{\varepsilon}{2} < \varepsilon, \]

which provides condition (iv) and concludes the proof.

Remark 2. A careful examination of the proof of Proposition 2 reveals that Condition D may be slightly relaxed without affecting the validity of Proposition 2. More precisely, let \( S \) be an infinite subset of \( \mathbb{N} \) and assume that \( \mathcal{U} \neq \emptyset \), where \( \mathcal{U} \) is now with respect to the elements \( \{x_j\}_{j \in S} \) of \( \mathcal{X} \). Then it suffices that Condition D is valid only for all finite subsets \( I \) of \( S \), but the indices \( j_n(i) \) may belong to \( \mathbb{N} \setminus S \).

Theorem 3. Let \( E \) be a topological vector space over \( \mathbb{K} \), the topology of which is induced by a translation invariant metric \( \rho \). Let \( \{x_j\}_{j \in \mathbb{N}} \) be a fixed sequence in \( E \) satisfying Condition D. Let \( \mathcal{X} \) denote the set of all finite linear combinations of the elements of the sequence \( \{x_j\}_{j \in \mathbb{N}} \) endowed with the metric \( \rho \). Then \( \mathcal{U} \neq \emptyset \) and there exists a scalar sequence \( a = \{a_j\}_{j \in \mathbb{N}} \) in \( \mathbb{K}^\mathbb{N} \), such that the sequence \( \{\sum_{j=0}^{n} a_j x_j\}_{n \in \mathbb{N}} \) is dense in \( \mathcal{X} \). The sequence \( a \) can be chosen in \( \cap_{p>1} \ell^p \). Moreover, the set of such sequences in \( A \) is a dense \( G_\delta \) in \( A = \cap_{p>1} \ell^p \) or \( \ell^q \), \( 1 < q < \infty \) or \( c_0 \) or \( \mathbb{K}^\mathbb{N} \), endowed with their natural topologies, and it contains a dense vector subspace except of zero.

Proof. Due to Proposition 2, it suffices to show that Condition D implies that \( \mathcal{U} \neq \emptyset \). Let \( x \in \mathcal{X} \) and \( m \in \mathbb{N} \). Then there exist \( M \in \mathbb{N} \) and \( \gamma_0, \ldots, \gamma_M \in \mathbb{K} \), such that \( x = \sum_{j=0}^{M} \gamma_j x_j \). According to Condition D, every \( x_j \) can be approximated by \( x_{j'} \)’s with \( j' > m \), which can be chosen so that \( \rho(\gamma_j x_j, \gamma_j x_{j'}) < \varepsilon / (m+1) \). It follows that

\[ \rho\left( x, \sum_{j=0}^{m} \gamma_j x_{j'} + \sum_{j=m+1}^{M} \gamma_j x_j \right) < \varepsilon. \]

Therefore, the linear span of \( x_{m+1}, x_{m+2}, \ldots \) is dense in \( \mathcal{X} \). Since this holds for every \( m \in \mathbb{N} \), it follows that \( \mathcal{U} \neq \emptyset \) ([GE99]). This completes the proof.

Remark 3. If in Theorem 3 the space \( \mathcal{X} \) is replaced by \( \overline{\mathcal{X}} \), the closure of \( \mathcal{X} \) in \( E \), or by \( \tilde{\mathcal{X}} \), the completion of \( \langle X, \rho \rangle \), then analogous results hold, since a sequence which is dense in \( \mathcal{X} \) is also dense in \( \overline{\mathcal{X}} \) and in \( \tilde{\mathcal{X}} \).

3. Examples

In this section we give examples where Theorem 3 applies. In one of these examples, the conclusion \( a \in \cap_{p>1} \ell^p \) can not be replaced by \( a \in \ell^1 \). In fact, the same is true in most of our examples.

Example 1. Let \( E = \mathbb{R} \) and set \( x_j = 1 \), for all \( j \in \mathbb{N} \). Then \( \mathcal{X} = \mathbb{R} \) and Condition D holds. Theorem 3 applied to this case yields the existence of a sequence \( \{a_j\}_{j \in \mathbb{N}} \) in \( \cap_{p>1} \ell^p \) so that the sequence of partial sums \( \{\sum_{j=0}^{n} a_j\}_{n \in \mathbb{N}} \) is dense in \( \mathbb{R} \).
Clearly, if \( a \in \ell^1 \), then the sequence \( \{ \sum_{j=0}^{n} a_j \}_{n \in \mathbb{N}} \), being convergent is not dense in \( \mathbb{R} \). Thus the condition \( a \in \cap_{p>1} \ell^p \) can not be replaced by \( a \in \ell^1 \). We reach the same conclusion by considering the case of any Banach space \( E \) where the sequence \( \{ x_j \}_{n \in \mathbb{N}} \) is bounded.

**Example 2.** Let \( \varphi : \mathbb{R}^v \to \mathbb{R} \) (or \( C \)) be a continuous function. The vector space \( E^1 \) is the set of continuous functions on \( \mathbb{R}^v \) endowed with several topologies, which shall be defined later. Let \( J = \{ b_j \}_{j \in \mathbb{N}} \) be an infinite subset of \( \mathbb{R}^v \) without isolated points. We define as \( x_j \) the function \( x_j(t) = \tau_{b_j} \varphi(t) = \varphi(t-b_j) \), \( j \in \mathbb{N} \). Let \( \mathcal{X} \) be the linear span of the \( x_j \)'s.

**Case A.** Let \( K \subset \mathbb{R}^v \) be a compact set. The set \( E \) is \( C(K) \) endowed with the supremum norm. The sequence \( \{ x_j \}_{j \in \mathbb{N}} \) satisfies Condition D since \( J \) does not contain isolated points and translation is continuous in \( C(K) \). Thus, there exists a \( a = \{ a_j \}_{j \in \mathbb{N}} \in \cap_{p>1} \ell^p \), for which the sequence of partial sums \( \{ \sum_{j=0}^{n} a_j \tau_{b_j} \varphi \}_{n \in \mathbb{N}} \) is dense in \( \mathcal{X} \) with respect to the uniform norm in \( K \).

**Case B.** We set \( E = E^1 = C(\mathbb{R}^v) \) with the standard distance yielding the topology of uniform convergence on compacta. The sequence \( \{ x_j \}_{j \in \mathbb{N}} \) satisfies Condition D, therefore there exists a sequence \( a = \{ a_j \}_{j \in \mathbb{N}} \in \cap_{p>1} \ell^p \), for which the sequence of partial sums \( \{ \sum_{j=0}^{n} a_j \tau_{b_j} \varphi \}_{n \in \mathbb{N}} \) is dense in \( \mathcal{X} \subset C(\mathbb{R}^v) \), which is the linear span of the \( x_j \)'s, with respect to the topology of the uniform convergence on compacta. An analogous example is obtained by replacing \( \mathbb{R}^v \) with \( \Omega \), an open subset of \( \mathbb{R}^v \), and \( E \) by \( C(\Omega) \) equipped with the topology of uniform convergence on the compact subsets of \( \Omega \).

**Case C.** Let \( \varphi \in C(\mathbb{R}^v) \) and assume that \( \lim_{t \to -\infty} \varphi(t) = 0 \). We set

\[
E = \{ u \in C(\mathbb{R}^v) : \text{such that } \lim_{t \to -\infty} u(t) = 0 \}.
\]

The space \( E \) becomes a Banach space when equipped with the supremum norm in \( \mathbb{R}^v \). Once again Theorem 3 applies. An analogous example is obtained if we assume that \( \varphi \) is periodic with \( v \) linearly independent periods \( T_1, \ldots, T_v \). Accordingly, the space \( E \) is defined as

\[
E = \{ u \in C(\mathbb{R}^v) : u(t + T_j) = u(t), \text{ for all } t \in \mathbb{R}^v \text{ and } j = 1, \ldots, v \},
\]
equipped with the supremum norm in \( \mathbb{R}^v \). A slightly more general example is when \( \varphi \) is uniformly continuous and bounded and \( E \) is the space of uniformly continuous and bounded functions in \( \mathbb{R}^v \) equipped with the supremum norm.

**Example 3.** Let \( \varphi : \mathbb{R}^v \to \mathbb{R} \) be a \( C^\infty \)-function and \( J = \{ b_j \}_{j \in \mathbb{N}} \) be an infinite subset of \( \mathbb{R}^v \) without isolated points. We define as \( x_j \) the function \( x_j(t) = \tau_{b_j} \varphi(t) = \varphi(t-b_j) \), \( j \in \mathbb{N} \). Let \( \mathcal{X} \) be the linear span of the \( x_j \)'s which we will endow with various topologies.
Case A. Let $\Omega \subset \mathbb{R}^v$ be open and $E = \cap_{t \in \mathbb{N}} C^t(\overline{\Omega})$. (A function $u$ belongs to $E$ if $u \in C^\infty(\Omega)$ and $D^a u$ extends continuously to the boundary of $\Omega$, for every multi–index $a$.) The space $E$ becomes a Fréchet space when equipped with a metric which is compatible with the topology of uniform convergence on the compact subsets of $\overline{\Omega}$, for every partial derivative. Condition D is clearly satisfied and Theorem 3 applies. Of particular interest are the cases $\Omega = \mathbb{R}^v$ and $\Omega$ bounded.

Case B. We further assume that $\lim_{t \to \infty} D^a \varphi(t) = 0$, for every multi–index $a$ or that $\varphi$ is periodic with $\nu$. We further assume that $\lim_{t \to \infty} \nu(t) = 0$, and define the set $J$ and the $\chi_j$’s as in Example 3. The space $X$ of linear combinations of the $\chi_j$’s is equipped with the $L^p$–norm. Theorem 3 again applies, since translations are continuous in $L^p(\mathbb{R}^v)$, $1 \leq p < \infty$.

Example 4. Let $\varphi \in L^p(\mathbb{R}^v)$, $1 \leq p < \infty$, and define the set $J$ and the $\chi_j$’s as in Example 3. The space $X$ of linear combinations of the $\chi_j$’s is equipped with the $L^p$–norm. Theorem 3 again applies, since translations are continuous in $L^p(\mathbb{R}^v)$, $1 \leq p < \infty$.

Example 5.

Sobolev spaces. Let now $\Omega$ be an open subset of $\mathbb{R}^v$, $k$ be a non–negative integer and $p \in [1, \infty)$. Assume that $\varphi \in W^{k,p}(\Omega)$, the Sobolev space of functions in $\Omega$ with weak derivatives of order up to $k$ in $L^p(\Omega)$. The set $J$ and the $\chi_j$’s are as in Example 3. The space $X$ of linear combinations of the $\chi_j$’s is equipped with the norm of the space $W^{k,p}(\Omega)$. Theorem 3 applies once again, since translations are continuous in $W^{k,p}(\Omega)$, whenever $k \in \mathbb{N}$ and $1 \leq p < \infty$. It is noteworthy that Theorem 3 does not apply in the case of the Sobolev spaces $W^{k,\infty}(\Omega)$. In this space Condition D does not hold. Also, translations are not continuous in $W^{k,\infty}(\Omega)$.

Hölder spaces. Let now $\sigma \in (0, 1)$ and $\Omega$ be an open bounded domain in $\mathbb{R}^v$. The space of Hölder functions $\Lambda^{0,\sigma}(\overline{\Omega})$ consists of all functions $u$, such that $|u|_\sigma = \sup_{|\delta| > 0} \omega_\sigma(u, \delta) < \infty$, where

$$\omega_\sigma(u, \delta) = \sup_{x, y \in \overline{\Omega}, 0 < |x - y| < \delta} \frac{|u(x) - u(y)|}{|x - y|^{\sigma}}.$$  

The space $\Lambda^{0,\sigma}(\overline{\Omega})$ is a Banach space with norm $|u|_{0,\sigma} = |u|_0 + |u|_\sigma$, where $|\cdot|_0$ is the norm of $C(\overline{\Omega})$. In general, if $k \in \mathbb{N}$, then the space $\Lambda^{k,\sigma}(\overline{\Omega})$ consists of all functions $u$ which, together with all their partial derivatives $D^a u$, $|\alpha| \leq k$, belong to $C^{0,\sigma}(\overline{\Omega})$. The space $\Lambda^{k,\sigma}(\overline{\Omega})$ is a Banach space with respect to the norm given by $|u|_{k,\sigma} = |u|_k + \max_{|\alpha| \leq k} |D^a u|_\sigma$. If $u \in \Lambda^{k,\sigma}(\overline{\Omega})$ and $\lim_{t \to 0} \omega_\sigma(u, \delta) = 0$, then $u$ is called uniformly Hölder continuous functions of order $(k, \sigma)$ in $\overline{\Omega}$. The set of uniformly Hölder continuous functions of order $(k, \sigma)$ in $\overline{\Omega}$, which is denoted by $C^{k,\sigma}(\overline{\Omega})$, is a closed subspace of $\Lambda^{k,\sigma}(\overline{\Omega})$ and thus a Banach space as well. Clearly, if $u \in C^{k,\sigma}(\overline{\Omega})$ can be approximated in the $|\cdot|_{k,\sigma}$–norm by functions which are $C^\infty$ in a neighborhood of $\overline{\Omega}$, then $u \in C^{k,\sigma}(\overline{\Omega})$. Thus, the elements of
\( \Lambda^{k,\sigma}(\overline{\Omega}) \setminus C^{k,\sigma}(\overline{\Omega}) \) can not be approximated by functions, which are \( C^\infty \) in a neighborhood of \( \overline{\Omega} \) ([Tar95, Theorem 1.3.6]).

Assume now that \( \varphi \in C^{k,\sigma}(\overline{\Omega}) \). It is readily shown that Condition D holds and also that translations are continuous in \( C^{k,\sigma}(\Omega) \), whenever \( k \in \mathbb{N} \) and \( \sigma \in (0,1) \). Let the set \( J \) and the \( x_j \)’s be as in Example 3. The space \( X \) of linear combinations of the \( x_j \)’s is equipped with the norm of the space \( C^{k,\sigma}(\overline{\Omega}) \). Theorem 3 applies once again. It is noteworthy that Theorem 3 does not apply if \( C^{k,\sigma}(\overline{\Omega}) \) is replaced by \( \Lambda^{k,\sigma}(\overline{\Omega}) \). In the latter space Condition D does not hold. Also, translations are not continuous in \( \Lambda^{k,\sigma}(\Omega) \).

**Remark 4.** In most of the previous cases \( \varphi \) is a continuous function defined in the whole \( \mathbb{R}^v \). However, we may also assume that \( \varphi \in C(V) \), where \( V = \{ x \in \mathbb{R}^v : \| x \| > r \} \), for some \( r \geq 0 \). Then, the subsets \( \Omega \) should be disjoint from \( J_r = \{ x \in \mathbb{R}^v : \text{dist}(x,J) \leq r \} \). Of special interest is the case \( r = 0 \) which occurs in some applications that follow. The difference between the previous examples and the applications that follow is that in the examples the space \( X \) is not in general explicitly identified, while in the applications, using some approximation results, we identify \( X \). Thus, we have universal series in \( \cap_{p>1} L^p \) realizing approximations in classical spaces such as \( C(\mathbb{R}) \), \( L^0(\mathbb{R}) \), \( \mathcal{H}(\Omega) \), the space of solutions of an elliptic operator in a given domain, \( L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), etc.

**4. Unification of Known Results**

In this section we present results from [NS07, NS, Steb] in a unified way as corollaries of Theorem 3.

The following is actually an extension of the results for restricted universal series obtained in [Steb]. Let \( J = \{ b_j \}_{j \in \mathbb{N}} \) be a countable subset of \( C \) with no isolated points satisfying \( \overline{J} \neq C \). Let, also, \( \varphi \) be a function of the form \( \varphi(z) = c/z + g(z) \), where \( c \) is a nonzero constant, and \( g \) is an entire function. In particular \( \varphi \) can be equal to \( 1/z \) or to the Riemann \( \zeta \)-function. We set \( x_j = \tau_{b_j} \varphi \). Let \( K \subset C \) be compact with connected complement and such that \( K \cap J = \emptyset \). As \( E \) we take the space \( A(K) \), that is the space of functions continuous on \( K \) and holomorphic in the interior \( K^0 \) of \( K \). It is readily seen that Condition D is satisfied, therefore there exists a sequence of complex numbers \( \{ a_j \}_{j \in \mathbb{N}} \) in \( \cap_{p>1} L^p \) so that the sequence \( \{ \sum_{j=1}^n a_j x_j \}_{n \in \mathbb{N}} \) is dense in the space \( X \) of finite linear combinations of \( \{ x_j \}_{j \in \mathbb{N}} \) equipped with the topology of uniform convergence on \( K \). Moreover, using the fact that \( \varphi \) is a fundamental solution of the \( \overline{\partial} \)-operator ([GT05, Steb]) combined with Mergelyan’s Theorem, one sees that the closure \( X \) of \( X \) in \( E \) is exactly \( A(K) \). In the particular case \( \varphi(z) = 1/z \), the same result is obtained by using an extension of Runge’s Theorem [LR84]. Let us now consider a collection of compact sets \( \{ K_k \}_{k \in \mathbb{N}} \) in \( C \) with the properties: (i) \( K_k \cap J = \emptyset \), (ii) \( C \setminus K_k \) is connected, and (iii) for every \( K \subset C \) compact with connected complement and \( K \cap J = \emptyset \) there exists \( k_0 \) such that \( K \subset K_{k_0} \). For the existence of such a collection we refer to [BGCMLO6], [GE87, Chapter 2.2], [Luh86, page 198]. The abstract theory of universal series provides us with a sequence \( \{ a_j \}_{j \in \mathbb{N}} \) in \( \cap_{p>1} L^p \) such that the sequence \( \{ \sum_{j=1}^n a_j x_j \}_{n \in \mathbb{N}} \) is dense in any one of the spaces \( A(K_k) \). Using Mergelyan’s
Theorem we automatically have density in every $\mathcal{A}(K)$, where $K$ is an arbitrary compact set with connected complement which does not intersect $\mathcal{J}$. Therefore, there exist universal series with $\cap_{p>1} l^p$ coefficients, which are universal simultaneously for every space $\mathcal{A}(K)$, where $K$ is a compact set with connected complement not intersecting $\mathcal{J}$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^v$ and let $J = \{b_j\}_{j \in \mathbb{N}}$ be a subset of $\mathbb{R}^v \setminus \overline{\Omega}$ without isolated points. Let $\mathcal{L}$ be an elliptic operator in $\mathbb{R}^v$ with constant coefficients of order $m$ and let $e = e(x)$ be a fundamental solution of $\mathcal{L}$ (i.e., $\mathcal{L}e = \delta$ in the sense of distributions).

We set $x_j = \tau_b e$. Clearly, $x_j$ is defined in $\mathbb{R}^v \setminus \{b_j\}$ and it is real analytic there. Let

$$E = \{u \in C^m(\Omega) : \mathcal{L}u = 0\} \cap C^K(\overline{\Omega}).$$

The space $E$ is a closed subspace of $C^K(\overline{\Omega})$. Clearly, Condition D is satisfied. Therefore, there exists a sequence of real numbers $a = \{a_j\}_{j \in \mathbb{N}}$ in $\cap_{p>1} l^p$, such that the partial sums $\{\sum_{j=0}^n a_j x_j\}_{n \in \mathbb{N}}$ are dense in $\mathcal{X}$, the linear span of the elements of $\{x_j\}_{j \in \mathbb{N}}$ equipped with the $C^K$-norm in $\overline{\Omega}$. We wish now to identify $\mathcal{X}$ in $E$. If the closure of $J$ in $\mathbb{R}^v$ contains an open set which intersects all the connected components of $\mathbb{R}^v \setminus \overline{\Omega}$ and $\Omega$ satisfies a mild boundary regularity condition, namely the Segment Condition$^1$, then $\overline{\mathcal{X}} = E$ ([Bro62, Smy06, Smy07, Wei73]). Thus, in such case, the partial sums $\{\sum_{j=0}^n a_j x_j\}_{n \in \mathbb{N}}$ are dense in $E$ ([NS07, Smy07]).

Next suppose that the closure of $J$ in $\mathbb{R}^v$, $v \geq 3$, is a pseudo–boundary which embraces$^2$ $\Omega$, the domain $\Omega$ still satisfies the Segment Condition and that $\mathcal{L} = -\Delta$. Then once again $\overline{\mathcal{X}} = E$, where

$$E = \{u \in C^2(\Omega) : \Delta u = 0\} \cap C^K(\overline{\Omega}),$$

and the partial sums $\{\sum_{j=0}^n a_j x_j\}_{n \in \mathbb{N}}$ are dense in $E$ ([NS07, Smy06, Smy07]).

Under the assumptions of the last case, we let $G$ be the component of $\mathbb{R}^v \setminus \overline{\mathcal{J}}$ containing $\Omega$, $E$ be the space of harmonic functions in $G$ equipped with the topology of the uniform convergence on the compact subsets of $G$. Again, $x_j$ is the function $\tau_b e$, where $e$ is a fundamental solution of $\mathcal{L} = -\Delta$. The Condition D is again satisfied. Therefore, there exists a sequence of real numbers $a = \{a_j\}_{j \in \mathbb{N}}$ in $\cap_{p>1} l^p$, such that the sequence of partial sums $\{\sum_{j=0}^n a_j x_j\}_{n \in \mathbb{N}}$ is dense in $\mathcal{X}$, the set of linear combinations of the $x_j$’s equipped with the topology of the uniform convergence on the compact subsets of $G$. Next, the closure $\overline{\mathcal{X}}$ of $\mathcal{X}$ in $E$ is the whole $E$ because $G$ has an exhausting family of compact sets $\{K_m\}_{m \in \mathbb{N}}$, so that $K_m = \overline{K_m^\circ}$. $K_m$ satisfies the Segment Condition and $\overline{\mathcal{J}}$ is a pseudo–boundary of $K_m$. Thus, in

$^1$**Definition.** (The Segment Condition) Let $\Omega$ be an open subset of $\mathbb{R}^v$. We say that $\Omega$ satisfies the segment condition if for every $x \in \partial \Omega$ there exists a neighborhood $U_x$ and a nonzero vector $\xi_x$ such that, if $y \in U_x \cap \Omega$, then $y + t \xi_x \in \Omega$ for every $t \in (0, 1)$.

Note that the Segment Condition allows the boundaries to have corners and cusps. Also, the boundary of the domains which satisfy this condition is $(v-1)$–dimensional. However, if a domain satisfies the segment condition it cannot lie on both sides of any part of its boundary. See [AF03].

$^2$**Definition.** (The Embracing Pseudo–boundary) Let $\Omega, \Omega'$ be open connected subsets of $\mathbb{R}^n$. We say that $\Omega'$ embraces $\Omega$ if $\overline{\Omega} \subset \Omega'$, and for every connected component $V$ of $\mathbb{R}^n \setminus \Omega$, there is an open connected component $V'$ of $\mathbb{R}^n \setminus \Omega'$ such that $\overline{V'} \subset V$. 

this case the partial sums \( \{ \sum_{j=0}^{n} a_j x_j \}_{n \in \mathbb{N}} \) are dense in the space of harmonic functions in \( G \) with respect to the topology of uniform convergence on the compact subsets of \( G \). This last result is not included in [NS07].

Finally, we turn to a result from [NS]. Let \( f = \{ b_j \}_{j \in \mathbb{N}} \) be countable and dense subset of \( \mathbb{R}^v \). Let \( \varphi_t, \varepsilon > 0 \) be an approximate identity in \( \mathbb{R}^v \) ([NS, SS05]). Let \( x_j = \tau b_j \varphi_{\varepsilon j}, c \) where \( (\varepsilon j, ij) \), \( j \in \mathbb{N} \) is an enumeration of \( \{ \frac{1}{\varepsilon} : n \in \mathbb{N}_+ \} \times \mathbb{N} \). The space \( E^1 \) is the linear span of the \( x_j \)'s and the continuous functions of compact support equipped with the norm \( \| \cdot \|_1 + \| \cdot \|_\infty \). Clearly, Condition D is satisfied. Thus, there exists a sequence of real numbers \( a = \{ a_j \}_{j \in \mathbb{N}} \) in \( \cap_{p>1} \ell^p \), such that the partial sums \( \{ \sum_{j=0}^{n} a_j x_j \}_{n \in \mathbb{N}} \) are dense in \( \mathcal{X} \), the linear span of the \( x_j \)'s equipped with the norm \( \| \cdot \|_1 + \| \cdot \|_\infty \). Now, the completion \( \overline{\mathcal{X}} \) contains the space of continuous functions with compact support. (This follows by using well-known theorems of approximate identities and in particular, by approximating a function \( f \in C_0(\mathbb{R}^v) \) by the convolution \( f * \varphi_t \), which in turn can be approximated by appropriate linear combinations of \( \tau b_j \varphi_{\varepsilon j} \) [NS].) Also, convergence in \( \| \cdot \|_1 + \| \cdot \|_\infty \) implies convergence in the \( L^p \)-norm, \( p \in [1, \infty] \), and almost everywhere convergence to measurable functions with respect to any \( \sigma \)-finite Borel measure. Thus, the universal series \( \sum_{j=0}^{\infty} a_j x_j \) constructed above, realize simultaneously all these approximations [NS]. Of particular interest is the case when \( \varphi_t \) are Gaussian laws, in which case the results of [NS] strengthen known results.

5. Universal trigonometric series in \( \mathbb{R}^v \)

Let \( E = C(\mathbb{R}) \), be equipped with the topology of uniform convergence on the compact subsets of \( \mathbb{R} \). Assume that the sequence \( \{ \beta_j \}_{j \in \mathbb{N}} \subset \mathbb{R} \) does not contain isolated points. Set \( x_j = x_j(t) = e^{i \beta_j t}, j \in \mathbb{N} \). Condition D is satisfied since every continuous function is uniformly continuous in every compact set and hence, translation of a continuous function is continuous with respect to the supremum norm in a compact set. In fact, it is readily seen that if \( K \) is compact and \( t \in K \subset [-M, M] \), then \( | e^{i \beta t} - e^{i \beta' t} | \leq M | \beta - \beta' | \). According to Theorem 3, there exist sequences \( a = \{ a_j \}_{j \in \mathbb{N}} \in \cap_{p>1} \ell^p \), such that the partial sums \( \{ \sum_{j=0}^{n} a_j e^{i \beta_j t} \}_{n \in \mathbb{N}} \) are dense in \( \mathcal{X} \), where \( \mathcal{X} \) is the set of linear combinations of the functions \( x_j(t) = e^{i \beta_j t}, j \in \mathbb{N} \), equipped with the topology of uniform convergence on compact sets. We shall show that \( \overline{\mathcal{X}} = C(\mathbb{R}) \). Indeed, if \( f \in C(\mathbb{R}) \) and \( K \subset \mathbb{R} \) compact, by Mergelyan’s Theorem \( f \) can be approximated uniformly on \( K \) by complex polynomials. Every such polynomial can be approximated by a linear combination of \( x_j \)'s because \( \{ i \beta_j \}_{j \in \mathbb{N}} \) has at least one accumulation point in \( \mathbb{C} \) and the linear span of the functions \( z \rightarrow e^{i \beta_j z}, j \in \mathbb{N} \) is dense in the set \( \mathcal{H}(\mathbb{C}) \) of entire functions endowed with the topology of uniform convergence on compact subsets of \( \mathbb{C} \) ([Lev96, Sha98]). Thus we have proved the following:

**Theorem 4.** Let \( \{ \beta_j \}_{j \in \mathbb{N}} \subset \mathbb{R} \) be a sequence without isolated points, then there exists a sequence \( \{ a_j \}_{j \in \mathbb{N}} \subset \mathbb{C} \) in \( \cap_{p>1} \ell^p \), so that, for every continuous function \( f : \mathbb{R} \rightarrow \mathbb{C} \), there exists a sequence \( \{ \lambda_n \}_{n \in \mathbb{N}} \subset \mathbb{N} \), such that \( \sum_{j=0}^{\lambda_n} a_j e^{i \beta_j t} \rightarrow f(t) \), uniformly in every compact subset.
of \( \mathbb{R} \), as \( n \to \infty \). The set of such sequences is a dense \( G_{\delta} \) in \( \cap_{p > 1} \ell^p \) equipped with its natural topology and contains a dense vector subspace except 0. Similar results hold if \( \cap_{p > 1} \ell^p \) is replaced by \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( C^\mathbb{N} \), equipped with their natural topologies.

\[ \sum \beta \]

\section*{Remark 5.} According to Remark 2, if \( \{ \beta_j \}_{j \in \mathbb{N}} \) contains a convergent sequence \( \{ \gamma_n \}_{n \in \mathbb{N}} \) of distinct elements and for each \( n \in \mathbb{N} \), there exists a sequence \( \{ \gamma_n \}_{n \in \mathbb{N}} \) also of distinct elements converging to \( \gamma_n \) as \( j \to \infty \), then the result of Theorem 4 still holds.

\section*{Remark 6.} We consider a sequence \( \Lambda = \{ \lambda_j \}_{j \in \mathbb{N}} \subset \mathbb{C} \) of distinct elements such that the linear combinations of the functions \( \{ e^{\lambda z} \}_{\lambda \in \Lambda} \) are dense in \( \mathcal{H}(C) \) (see [Lev96, Sha98]). Removing a finite subset from \( \Lambda \), the above fact remains valid, therefore, there exist unrestricted universal series realizing approximations in \( \mathcal{H}(C) \) and consequently \( \mathcal{U} \neq \emptyset \). If now we make the stronger assumption that \( \Lambda \) does not possess isolated points, then we shall have that \( \mathcal{U}_{p > 1} \ell^p \neq \emptyset \); hence these universal series realize approximations in the space \( \mathcal{H}(C) \) of entire functions.

We wish to extend Theorem 4 to \( \mathbb{R}^v \). We consider \( J^1, \ldots, J^v \) infinite countable subsets of \( \mathbb{R} \) without isolated points. Let \( \beta_j = (\beta_j^1, \ldots, \beta_j^v), j \in \mathbb{N} \) an enumeration of \( J^1 \times \cdots \times J^v \). We let \( x_j : \mathbb{R}^v \to \mathbb{C} \) be defined as

\[ x_j(t) = e^{\beta_j t} = e^{\beta_j^1 t_1 + \cdots + \beta_j^v t_v}, \]

where \( t = (t_1, \ldots, t_v) \in \mathbb{R}^v \). Let \( E = C(\mathbb{R}^v) \) be equipped with the topology of uniform convergence on compact sets of \( \mathbb{R}^v \). Obviously, Condition D is satisfied and Theorem 3 gives a sequence of complex numbers \( a = \{ a_j \}_{j \in \mathbb{N}} \in \cap_{p > 1} \ell^p \), such that the partial sums \( \{ \sum_{j=0}^n a_j x_j \}_{n \in \mathbb{N}} \) are dense in \( \mathcal{X} \), where \( \mathcal{X} \) is the set of linear combinations of the functions \( e^{\beta_j t}, j \in \mathbb{N} \). We have to show that this last linear span is dense in \( C(\mathbb{R}^v) \). Let \( f \in C(\mathbb{R}^v), \epsilon > 0 \), and \( K \subset \mathbb{R}^v \) be compact. Due to the Stone–Weierstrass Theorem, there exists a function of the form \( \tilde{f}(t_1, \ldots, t_v) = \sum_{\mu=1}^m \prod_{l=1}^v h_{\mu,l} \), where \( h_{\mu,l} \) are continuous functions of one real variable so that \( \sup_{t \in K} |f(t) - \tilde{f}(t)| < \epsilon \). Now, according to Theorem 4, each \( h_{\mu,l}(t_l) \) may be uniformly approximated on a sufficiently large compact interval of \( t_l \) by a function of the form \( \sum_{j=0}^{l_{\mu,l}} c_{\mu,l,j} e^{i \gamma_j t_l} \), with \( \gamma_j \in J^l, \mu = 1, \ldots, m, l = 1, \ldots, v \). This fact combined with the boundedness of each continuous function on each compact set implies the result. We have proved the following:

\section*{Theorem 5.} We consider \( J^1, \ldots, J^v \) infinitely countable subsets of \( \mathbb{R} \) without isolated points. Let \( \beta_j = (\beta_j^1, \ldots, \beta_j^v), j \in \mathbb{N} \) an enumeration of \( J^1 \times \cdots \times J^v \). Then, there exists a sequence of complex numbers \( a = \{ a_j \}_{j \in \mathbb{N}} \in \cap_{p > 1} \ell^p \), so that for every complex continuous function \( f \) in \( \mathbb{R}^v \), there exists a sequence \( \{ \lambda_n \}_{n \in \mathbb{N}} \subset \mathbb{N} \), so that \( \sum_{j=0}^n a_j e^{i \beta_j t} \to f(t) \) uniformly on compact sets of \( \mathbb{R}^v \), as \( n \to \infty \). The set of such sequences \( a \) is a dense \( G_{\delta} \) in \( \cap_{p > 1} \ell^p \), and contains a dense vector space except 0. Similar results hold, if \( \cap_{p > 1} \ell^p \) is replaced by \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( C^\mathbb{N} \). □

If \( \mu \) is a positive \( \sigma \)–finite Borel measure on \( \mathbb{R}^v \) and \( g \) is a complex \( \mu \)–measurable function on \( \mathbb{R}^v \), then by Lusin’s Theorem, there exists a sequence of continuous functions \( f_n : \mathbb{R}^v \to \)
C, \( n \in \mathbb{N} \), and a Borel set \( E \subset \mathbb{R}^v \) with \( \mu(\mathbb{R}^v \setminus E) = 0 \), such that \( f_n(t) \to g(t) \) as \( n \to \infty \), for all \( t \in E \). Now, if \( a \) is a sequence given by Theorem 5, we can find a strictly increasing sequence \( \{r_n\}_{n \in \mathbb{N}} \subset \mathbb{N} \), so that \( \sup_{t < r_n} |f_n(t) - \sum_{j=0}^{r_n} a_j e^{i \beta_j t}| < 1/n \). It follows that \( \sum_{j=0}^{r_n} a_j e^{i \beta_j t} \to g(t) \), as \( n \to \infty \), for all \( t \in E \). We have proved the following:

**Corollary 1.** Let \( \{\beta_j\}_{j \in \mathbb{N}} \) be as in Theorem 5 and let \( a = \{a_j\}_{j \in \mathbb{N}} \) be a sequence given by Theorem 5. Let \( \mu \) be a \( \sigma \)-finite Borel measure in \( \mathbb{R}^v \). Then for every \( \mu \)-measurable function \( h : \mathbb{R}^v \to \mathbb{C} \) there exists a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{N} \), so that \( \sum_{j=0}^{\lambda_n} a_j e^{i \beta_j t} \to h(t) \), as \( n \to \infty \), \( \mu \)-almost everywhere.

It is noteworthy, that taking the real part of the formal series \( \sum_{j=0}^{\infty} a_j e^{i \beta_j t} \) we obtain a real universal series.

6. **Universal Dirichlet series**

The results of Section 5 can be improved if we use universal Dirichlet series ([Bay05]). We obtain the same results using only the frequencies \( \{\log j\}_{j \geq b} \).

**Definition 2.** ([Bay05]) A compact set \( K \) is said to be admissible for Dirichlet series if \( \mathbb{C} \setminus K \) is connected and if it can be written as \( K = \bigcup_{j=1}^m K_j \), with each \( K_j \) contained in a strip \( S_j = \{z \in \mathbb{C} : \Re z \in (a_j, b_j)\} \), with \( b_j - a_j < 1/2 \) and all the strips being disjoint.

A sequence \( a = \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \) belongs to the class \( D_a(C_+) \) if the Dirichlet series \( \sum_{j=1}^{\infty} a_j j^{-z} \) is absolutely convergent for every \( z \) in the open right-half plane \( C_+ = \{z \in \mathbb{C} : \Re z > 0\} \); that is \( \sum_{j=1}^{\infty} |a_j| j^{-\sigma} \to 0 \), for every \( \sigma > 0 \). The space \( D_a(C_+) \) is equipped with a metric \( d_1 \) which is compatible with the norms \( |a|_\sigma = \sum_{j=1}^{\infty} |a_j| j^{-\sigma} \), \( \sigma > 0 \). It is not difficult to see that \( D_a(C_+) \) satisfies the postulates \( P_1 - P_4 \) of Section 2.

**Theorem 6.** ([Bay05]) There exists a sequence \( a = \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \) which belongs to the class \( D_a(C_+) \), so that the partial sums \( S_n(z) = \sum_{j=1}^{n} a_j j^{-z} \), \( n \in \mathbb{N} \), satisfy the following: For every admissible compact set \( K \subset \mathbb{T}_- = \{z \in \mathbb{C} : \Re z \leq 0\} \), and for every complex function \( g \) which is continuous on \( K \) and holomorphic in \( K^c \), there exists a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{N} \), such that \( \sup_{z \in K} |S_{\lambda_n}(z) - g(z)| \to 0 \), as \( n \to \infty \). The set of such sequences is a dense \( G_\delta \) in \( D_a(C_+) \) and contains a dense vector subspace except 0.

Furthermore, it is noticed in [Bay05] that there exists a sequence \( a = \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \) satisfying the requirements of Theorem 6, such that \( a_j = o(j^{-r}) \), for every \( r < 1 \).

**Claim.** Such a sequence \( a = \{a_j\}_{j \in \mathbb{N}} \) belongs to \( \cap_{p > 1} \ell^p \).

**Proof of the Claim.** Let \( p > 1 \) be fixed; then \( \frac{1}{p} < 1 \) and we may choose \( r \) so that \( \frac{1}{p} < r < 1 \). We have \( |a_j| \leq j^{-r} \), for all \( j \) large enough. This implies that \( |a_j|^p \leq j^{pr} \) and since \( pr > 1 \) it follows that \( a \in \ell^p \). Since this holds for any \( p > 1 \) the claim is established.

We denote by \( d_2 \) the metric in \( \cap_{p > 1} \ell^p \) which is compatible with the norms of \( \ell^p \), \( p > 1 \), as in the proof of Proposition 2. We equip the space \( D_a(C_+) \cap (\cap_{p > 1} \ell^p) \) with the metric \( d_1 + d_2 \). Clearly, this space satisfies the postulates \( P_1 - P_4 \) and the abstract theory of universal
The metric of this space is the sum of a metric where

\[ \sum_{j=1}^{\infty} a_j j^{-\nu} \]

for all \( \nu \). Let \( K \subset \mathbb{C}_- \) be an admissible compact set in \( \mathbb{C}_- \) which belongs to both \( D_a(C_+) \) and \( \cap_{p>1} \ell^p \), with the property that the partial sums \( S_n(z) = \sum_{j=1}^{n} a_j j^{-\nu} \), \( n \geq 1 \), satisfy the following:

For every admissible compact set \( K \subset \mathbb{C}_- = \{ z \in \mathbb{C} : \text{Re} z \leq 0 \} \) and for every complex function \( g \) continuous on \( K \) and holomorphic in \( K^o \), there exists a sequence \( \{ \lambda_n \}_{n \in \mathbb{N}} \), such that \( \sup_{z \in K} |S_n(z) - g(z)| \to 0 \), as \( n \to \infty \). The set of such sequences \( a \) is a dense \( G_\delta \) in \( D_a(C_+) \cap (\cap_{p>1} \ell^p) \) and contains a dense vector space except 0. Similar results can be obtained if we replace \( \cap_{p>1} \ell^p \) with \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \), or if we replace \( D_a(C_+) \cap (\cap_{p>1} \ell^p) \) with \( \cap_{p>1} \ell^p \), \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \).

If we apply Theorem 7 in the case \( K \subset i\mathbb{R} \), then we conclude that the partial sums of the series \( \sum_{j=1}^{\infty} a_j j^{-t} = \sum_{j=1}^{\infty} a_j e^{-i(\log j)t} \), \( t \in \mathbb{R} \), are dense in the space \( C(\mathbb{R}) \) of continuous functions \( f : \mathbb{R} \to \mathbb{C} \), equipped with the topology of uniform convergence on compact sets of \( \mathbb{R} \). Since by modifying a finite number of terms in a universal series we once again obtain a universal series, we assume that \( a_j = 0 \) for \( \log j \leq \beta \), where \( \beta \) is an arbitrary real number. Furthermore, by changing \( t \) to \(-x\) we obtain strengthened versions of Theorem 4.

Next we investigate how to extend the above result in the case of several variables; this leads to considering universal Dirichlet series in several complex variables, a notion not well defined. Let \( K^1, \ldots, K^v \) be admissible compact subsets of \( \mathbb{C}_- \) and \( K = K^1 \times \cdots \times K^v \subset (\mathbb{C}_-)^v \). The space \( \mathcal{X}(K) \) contains the finite sums of products \( f^1(z_1) \cdots f^v(z_v) \), where each \( f^i \) is a complex function continuous on \( K^i \) and holomorphic in \( (K^i)^o \). We endow this space with the topology of uniform convergence on \( K \). Let \( \{ x_i \}_{i \in \mathbb{N}} \) be an enumeration of the functions of the form \( j_1^{-z} \cdots j_v^{-z} \), i.e.,

\[ x_i(z_1, \ldots, z_v) = j_1(i)^{-z_1} \cdots j_v(i)^{-z_v}, \quad i \in \mathbb{N}. \]

We shall consider sequences \( a = \{ a_j \}_{j \in \mathbb{N}} \subset \mathbb{C} \), which belong to \( \cap_{p>1} \ell^p \) and \( D_a(C^v_+) \), where

\[ D_a(C^v_+) = \left\{ a = \{ a_j \}_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : \sum_{j \in \mathbb{N}} |a_j| j_1(i)^{-\sigma_1} \cdots j_v(i)^{-\sigma_v} < \infty \quad \text{for all} \quad \sigma_1, \ldots, \sigma_v > 0 \right\}. \]

The metric of this space is the sum of a metric \( d_2 \) of the space \( \cap_{p>1} \ell^p \) and of a metric \( d_3 \) of the space \( D_a(C^v_+) \) which is compatible with the norms \( |a|_{\sigma_1, \ldots, \sigma_v} = \sum_{j \in \mathbb{N}} |a_j| j_1(i)^{-\sigma_1} \cdots j_v(i)^{-\sigma_v} \), for all \( \sigma_1, \ldots, \sigma_v > 0 \).

**Theorem 8.** There exists a sequence \( a = \{ a_j \}_{j \in \mathbb{N}} \subset \mathbb{C} \), which belongs to both \( D_a(C^v_+) \) and \( \cap_{p>1} \ell^p \), with the property that the partial sums \( S_n(z_1, \ldots, z_v) = \sum_{j=1}^{n} a_j j_1(i)^{-z_1} \cdots j_v(i)^{-z_v} \), \( n \geq 1 \), satisfy the following: For every compact set \( K \) of the form \( K = K^1 \times \cdots \times K^v \), where the \( K^i \)'s are admissible compact sets included in \( \mathbb{C}_- \), and for every function \( g \in \mathcal{X}(K) \), there exists a sequence \( \{ \lambda_n \}_{n \in \mathbb{N}} \subset \mathbb{N} \), such that \( \sup_{z \in K} |S_n(z_1, \ldots, z_v) - g(z_1, \ldots, z_v)| \to 0 \), as \( n \to \infty \),

series ([BGENP, NP05]) can be applied since there exists a class \( \{ K_1 \}_{j \in \mathbb{N}} \) of admissible compact sets in \( \mathbb{C}_- \) which absorbs every admissible compact set in \( \mathbb{C}_- \) ([Bay05, DM07]). Thus, we obtain the following:

**Theorem 7.** There exists a sequence \( a = \{ a_j \}_{j \in \mathbb{N}} \subset \mathbb{C} \), which belongs to both \( D_a(C_+) \) and \( \cap_{p>1} \ell^p \), with the property that the partial sums \( S_n(z) = \sum_{j=1}^{\infty} a_j j^{-\nu} \), \( n \geq 1 \), satisfy the following: For every admissible compact set \( K \subset \mathbb{C}_- \) which belongs to both \( D_a(C_+) \) and \( \cap_{p>1} \ell^p \) and contains a dense vector space except 0. Similar results can be obtained if we replace \( \cap_{p>1} \ell^p \) with \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \), or if we replace \( D_a(C_+) \cap (\cap_{p>1} \ell^p) \) with \( \cap_{p>1} \ell^p \), \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \).
uniformly on \( K \). The set of such sequences \( \alpha \) is a dense \( G_\delta \) in \( \cap_{p>1} \ell^p \cap D_\alpha (\mathbb{C}_p^+) \) and contains a dense vector space except 0. We have similar results if we replace \( \cap_{p>1} \ell^p \) with \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \), or if we replace \( \cap_{p>1} \ell^p \cap D_\alpha (\mathbb{C}_p^+) \) with \( \cap_{p>1} \ell^p , \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \).

Sketch of the proof. It is possible to extract a countable absorbing family of such compact sets \( K \), and thus, by using Mergelyan’s Theorem, we can apply the abstract theory of universal series ([BGENP, NP05]). It suffices to consider a compact set \( K = k^1 \times \cdots \times k^v \) and a function \( f \) in \( \mathcal{X}(K) \). We have to approximate \( f \) in \( K \) by a finite linear combination of \( \lambda \)'s, such that the sequence of coefficients is close to 0 in the metric \( d_2 + d_3 \). But \( f \) is a finite sum of products of the form \( f^1(z_1) \cdots f^v(z_v) \). Certainly, each of these functions is bounded. Furthermore, the functions \( f^1, \ldots, f^v \) can be approximated by finite linear combinations of the form \( c_{i,j} j_1^{-z_1} + \cdots + c_{i,j} j_\ell^{-z_\ell} \), with the sequence of coefficients to be close to 0 in the metric \( d_1 + d_2 \), because of the existence of universal Dirichlet series in one variable (Theorem 7). The result now follows from simple computations.

If \( K = k^1 \times \cdots \times k^v \) is such that \( k^1, \ldots, k^v \subset \mathbb{R} \), then by the Stone–Weierstrass Theorem, the space \( \mathcal{X}(K) \) is dense in \( C(K) \). Thus, Theorem 6.4 implies the following results.

**Theorem 9.** There exists a sequence \( \alpha = \{a_j\}_{j \in \mathbb{N}} \subset \mathbb{C} \), which belongs to both \( D_\alpha (\mathbb{C}_p^+) \) and \( \cap_{p>1} \ell^p \), with the property that the partial sums \( S_n(z_1, \ldots, z_v) = \sum_{k=1}^n a_k j^1(i)^{-z_1} \cdots j^v(i)^{-z_v} \), \( n \geq 1 \), are dense in \( C((\mathbb{R})^v) \) endowed with the topology of uniform convergence on the compact subsets. The set of such sequences \( \alpha \) is a dense \( G_\delta \) in \( \cap_{p>1} \ell^p \cap D_\alpha (\mathbb{C}_p^+) \) and contains a dense vector space except 0. We have similar results if we replace \( \cap_{p>1} \ell^p \) with \( \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \), or if we replace \( \cap_{p>1} \ell^p \cap D_\alpha (\mathbb{C}_p^+) \) with \( \cap_{p>1} \ell^p , \ell^q \), \( 1 < q < \infty \), \( c_0 \) or \( \mathbb{C}^N \).

The sequences given by Theorem 9 satisfy the analogue of Corollary 1 for all \( \sigma \)-finite Borel measures in \( (\mathbb{R})^v \), which is isomorphic to \( \mathbb{R}^v \). The question that remains open is what could be an appropriate definition of universal Dirichlet series in several complex variables.

**ACKNOWLEDGEMENTS**

The authors wish to thank G. Costakis and D. Gatzouras for the illuminating conversations they had with them.

**REFERENCES**


DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CYPRUS, 1678 NICOSIA, CYPRUS
E-mail address: skoumand@ucy.ac.cy

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ATHENS, PANEPISTIMIOPOLIS, 15784 ATHENS, GREECE
E-mail address: vnestor@math.uoa.gr

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CYPRUS, 1678 NICOSIA, CYPRUS
E-mail address: smyrlis@ucy.ac.cy

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF CYPRUS, 1678 NICOSIA, CYPRUS
E-mail address: vstefan@ucy.ac.cy