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Undergraduate Thesis

Supersymmetric QCD with overlap fermions

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Abstract

Constructing theories Beyond the Standard Model (BSM) can help us explain any shortcomings of the Standard Model, such as the nature of dark energy and dark matter or the matter-antimatter antisymmetry. Supersymmetric models of strongly coupled theories, such as supersymmetric extensions of Quantum Chromodynamics (QCD) on the lattice, are very promising. The study of SQCD, as a prototype model for complex supersymmetric theories, can be a valuable tool to explore how Supersymmetry (SUSY) might manifest itself in nature and potentially impact high energy particle physics.

However, since Supersymmetry is broken explicitly within the lattice framework, several challenges come up, including the fine-tuning of the parameters of the bare Lagrangian. During the renormalization of the theory, we would like to use a lattice discretization that exploits as many as possible of the continuum symmetries. In this work, we use the overlap formulation for fermions to preserve chiral symmetry in an $\mathcal{N} = 1$ version of SQCD and "prepare the ground" for the computation of renormalization coefficients and Green functions at any order of the coupling constant.

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Everything is relative
Distance, space, and time
But even when we are apart
You are absolutely mine.
-Q's

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Chapter 1

Introduction to Quantum Field Theory

1.1 Fields and the Necessity of the Field Viewpoint

Fields are defined as physical quantities taking values at each space-time point. According to their behavior under rotations, they can be scalar, vector, spinor, or tensor.

A classical example for a scalar field is the temperature, which is described by a number for each point in space and time. The electric and magnetic field correspond to a vector in every space-time point, and together they make up the electromagnetic field, which can be described by a rank-2 tensor.

Quantum Field Theory (QFT) is concerned with the quantization of dynamical systems of relativistic fields, just as in quantum mechanics dynamical systems of non-relativistic particles are quantized. But why do we need to quantize *fields* in order to understand processes in small (quantum-mechanical) scales and large (relativistic) energies? Why is it not possible to quantize relativistic particles in the same way we quantize non-relativistic particles?

As a first approach, we can work with the Klein-Gordon equation or the Dirac equation, which describe a single relativistic particle. In this case, negative energies and other inconsistencies arise. This is due to the fact that relativistic processes, which follow the Einstein relation, $E = mc^2$, cannot be explained in terms of a single particle, since they allow the creation of pairs of particles-antiparticles. Even if the initial energy is insufficient to create a particle-antiparticle pair, multiparticle states may still appear. These states can exist for a really short time, as a result of Heisenberg's Uncertainty Principle ($\Delta E \cdot \Delta t = \hbar$). In higher orders of perturbation theory, we can have the creation of arbitrarily many of these "virtual" states [10].

A multiparticle theory is also the indirect result of causality [10]. For a free particle to propagate from $\mathbf{x}_0$ to $\mathbf{x}$, the amplitude is:

$$U(t) = \langle \mathbf{x} | \, e^{-iHt} \, | \mathbf{x}_0 \rangle \, . \tag{1.1}$$

The energy in non-relativistic quantum mechanics is $E = \mathbf{p}^2/2m$, and we have

$$\begin{aligned} U(t) &= \langle \mathbf{x} | \, e^{-i(\mathbf{p}^2/2m)t} \, | \mathbf{x}_0 \rangle \\ &= \int d^3p \, \langle \mathbf{x} | \, e^{-i(\mathbf{p}^2/2m)t} \, | \mathbf{p} \rangle \, \langle \mathbf{p} | \mathbf{x}_0 \rangle \\ &= \frac{1}{(2\pi)^3} \int d^3p e^{-i(\mathbf{p}^2/2m)t} \cdot e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} \\ &= (\frac{m}{2\pi i t})^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t}. \end{aligned} \tag{1.2}$$

This expression shows that in arbitrarily short time, the particle can propagate between any two points in space, since the expression is non-zero for any $\mathbf{x}$ and t. This goes against the relativistic principle of causality. Even if we use the relativistic expression for energy, $E = \sqrt{p^2 + m^2}$, we will find that the result $U(t) \sim e^{-m\sqrt{\mathbf{x}^2 - t^2}}$ still violates causality, since it is non-zero outside the light-cone.

Quantum Field Theory provides a solution to the causality problem, because in this multiparticle field theory, there is no difference between a particle propagating across a spacelike integral and an antiparticle propagating in the opposite direction. The amplitudes of a propagating particle and a propagating antiparticle, when making an observation at $\mathbf{x}_0$ that could affect an observation made at $\mathbf{x}$, cancel each other, and causality is not violated.

The importance of Quantum Field Theory comes up not only when discussing ways to naturally handle multiparticle states but also transitions between states with different particle numbers. With the introduction of antiparticles, the causality problem is solved and we can find an explanation for the relation between spin and statistics as well. However, QFT really shines because of the tools it provides for the calculation of observable quantities, such as particle lifetimes and cross sections, which can be measured experimentally with precision and confirm any further predictions [10].

1.2 The Path Integral Approach to Quantization

The Path Integral Method in Quantum Mechanics

In Quantum Mechanics, the states of a system are described by vectors in a Hilbert Space, and observable quantities are expressed as hermitian operators acting upon

them. The Schrödinger equation provides the time evolution of a system, which can also be expressed as

$$|\psi(t)\rangle = e^{-iH(t-t_0)}\,|\psi(t_0)\rangle\,, \tag{1.3}$$

where H is the Hamiltonian of the system. We can therefore find the state of the system in a later time t, $|\psi(t)\rangle$, just by knowing its initial state, $|\psi(t_0)\rangle$.

Let $q = \{q_\alpha\}$ express the collective coordinate degrees of freedom of a system with simultaneous eigenstates $|q\rangle$ that correspond to the operators $\{Q_\alpha\}$:

$$Q_\alpha\,|q\rangle = q_\alpha\,|q\rangle\,, \quad \alpha = 1, ..., n$$

The wavefunction $\psi(q,t) = \langle q|\psi(t)\rangle$ following (1.3), hence satisfies the equation

$$\psi(q',t') = \int \mathrm{d}q\; G(q',t';q,t)\psi(q,t)$$

where

$$G(q',t';q,t) = \langle q'|\, e^{-iH(t'-t)}\,|q\rangle \tag{1.4}$$

is the Green function that describes the propagation of the state $|\psi(t)\rangle$, and dq is the integration measure and it is expressed as

$$\mathrm{d}q = \prod_{\alpha=1}^{n} \mathrm{d}q_\alpha$$

The Green function follows the composition law:

$$G(q',t';q,t) = \int \mathrm{d}q''\; G(q',t';q'',t'')G(q'',t'';q,t). \tag{1.5}$$

Using this property of the Green Function, Feynman expressed the matrix element (1.4) as a path integral (integrating over all possible paths that start and end in specified space-time points). The Green Function clearly illustrates how classical and quantum theories are connected. Classically, the time evolution of a system is given by the principle of least action and the equations Euler-Lagrange. The canonical quantization of the system is the result of finding the Hamiltonian and writing in Poisson brackets the equations of motion, but the connection with Hamilton's principle is lost. Feynman's path integral representation however, makes more transparent the relation between Hamilton's principle and the quantization of the system [11].

By discretizing the time interval to infinitesimal small time slices for a Hamiltonian of the form $H(Q,P) = T(P) + V(Q)$, where $T(P)$ is a polynomial in the canonical

momenta, and calculating the gaussian integrals that arise over all possible values of the coordinates at intermediate time steps that start at point q and end at point q' (that is, over all the possible paths that connect the two space-time points), one arrives at the following expression for the Green Function [6], [11]

$$\langle q'| \, e^{-iH(t'-t)} \, |q\rangle = N \int_q^{q'} Dq \; e^{iS[q]}, \tag{1.6}$$

where $S[q]$ is the action for real time, N is a normalization factor and

$$Dq = \lim_{n\to\infty} \prod_{i=1}^{n} \mathrm{d}q(t_i). \tag{1.7}$$

In Quantum Mechanics, as well as in Quantum Field Theory, large families of Green functions can be evaluated in Euclidean, rather than Minkowski, space, by means of a "Wick rotation", in which the time integration variable runs over the imaginary axis [10]. In Euclidean space, the paths would be weighted with $\exp(-S_E)$, and the paths close to the classical euclidean one (where $\delta S_E[q] = 0$), are expected to provide the dominant contribution. Therefore, in the path integral representation, a classical system is quantized by considering fluctuations around the classical path. In euclidean space, these fluctuations are exponentially suppressed if $S_E \geq 0$, while in real time formulation the paths are weighted with an oscillating function. For this reason, the real time formulation of the path integral, in which some contributions will cancel out, cannot be applied for numerical calculations [11].

Path Integral Representation in Field Theory

The Green function (1.4) contains all the physical information needed for a quantum-mechanical system. On the contrary, in field theory, this information is given by an infinite set of vacuum expectation values of products of field operators ($\phi(x) = \phi(\mathbf{x}, t)$), that are also time-ordered. The system's infinite number of coordinate degrees of freedom are denoted by the vector $\mathbf{x}$ and x is hence a four-vector that describes space-time.

In QFT we define the Green function of a field ϕ as:

$$G(x_1, x_2, ..., x_l) = \langle \Omega | \, T(\phi(x_1)\phi(x_2)...\phi(x_l)) \, |\Omega\rangle \, , \tag{1.8}$$

where $x_i = (\mathbf{x}_i, t_i)$ and $|\Omega\rangle$ is the vacuum state of the system, corresponding to a given Hamiltonian (also known as the ground state of the system). T is the time-order operator that puts the operators in descending time from left to right.

By following a similar framework as in the case of Quantum Mechanics [11], we can calculate the Green function of observables as

$$G(x_1, x_2, ..., x_l) = \frac{\int D\phi\ \phi(x_1)\phi(x_2)...\phi(x_l)e^{iS[\phi]}}{\int D\phi\ e^{iS[\phi]}} \tag{1.9}$$

where $\int D\phi$ is the sum over all possible configurations of $\phi(x)$ and $S[\phi]$ is the action that describes the system. Quantum fluctuations lead to effects contained in contributions to the integral (eq. (1.9)) that come from field configurations which are not solutions to the classical equation of motion and hence do not correspond to a stationary action.

The Green function of the product of two fields is the propagator from an initial to a final configuration. Higher order Green functions do not have a direct physical meaning but by calculating them, we have all the physical information that we need for the dynamical system.

Now, to be able to do any calculations, as mentioned before, we need to move from Minkowski space-time to Euclidean space-time, namely to go from real to imaginary time ($t \to -i\tau$) [11]. By doing this substitution, we also have

$$iS[q] \xrightarrow{"t\to -i\tau"} -S_E[q].$$

Then equation (1.9) becomes:

$$G(x, y, ...) = \frac{\int D\phi\ \phi(x)\phi(y)...e^{-S_E[\phi]}}{\int D\phi\ e^{-S_E[\phi]}} \tag{1.10}$$

where x and y are now four-vectors in Euclidean space-time, and the Green function takes the form of a correlation function of a statistical mechanical system. The paths are then weighted by a "Boltzmann factor", $\exp(-S_E)$ [11].

Equation (1.10) only holds in the case where the fields ϕ are scalar (bosonic fields). By following a similar methodology as before [11], we can find the expression for Green functions of fermionic fields

$$\langle \psi_\alpha(x)...\bar{\psi}_\beta(y)...\rangle = \frac{\int D\bar{\psi}D\psi\ (\psi_\alpha(x)...\bar{\psi}_\beta(y)...)e^{-S_E[\psi,\bar{\psi}]}}{\int D\bar{\psi}D\psi\ e^{-S_E[\psi,\bar{\psi}]}}, \tag{1.11}$$

where ψ and $\bar{\psi}$ are hermitian conjugates of each other and $S_E[\psi, \bar{\psi}]$ is the corresponding euclidean action. Since ψ and $\bar{\psi}$ represent fermionic fields that anticommute in nature when $\hbar \to 0$, they become elements of the Grassmann algebra in this limit (see properties of Grassmann variables in Appendix A).

An exact calculation of these Green functions is only possible in very few cases, mostly when the action has linear and quadratic terms. Even the example of the Coulomb potential is non-trivial (Duru and Kleinert, 1979). In quantum mechanics, there are more efficient ways than the path integral to calculate eigenstates and eigenvalues or scattering amplitudes of a system. But in Quantum Field Theory, when higher order terms appear in the action (interaction terms for example), we can use perturbation theory in order to compute the necessary Green functions and in this case, the path integral method is really useful. Using path integrals, the Feynman rules and diagrams of a gauge theory can be easily derived [10], [11].

Chapter 2

Quantum Field Theory in the Continuum

2.1 Klein-Gordon Equation for free scalar fields

The Klein-Gordon equation is a classical field equation that describes a real free field ϕ:

$$(\Box + m^2)\phi(x) = 0, \tag{2.1}$$

where $\Box$ is the d'Alembert operator, x is a space-time four-vector $x_\mu = (x^0, x^1, x^2, x^3)$ and m is the mass of a particle with spin 0. This equation is a result of applying Hamilton's principle ($\delta S = 0$) to the action:

$$S = -\frac{1}{2}\int d^4x \phi(x)(\Box + m^2)\phi(x) \tag{2.2}$$

that arises from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

.

The Klein-Gordon equation is a relativistic equation; it is a second order partial differential equation both in space and time, as relativity dictates. It was rejected because of some problems that come up due to the second order time-dependence, such as two initial conditions, infinite negative energies and negative probability densities. In its modern interpretation, the equation describes a classical multiparticle theory of a free complex scalar field (the amplitude contributed by a particle and an antiparticle propagating in opposite directions is zero, thus preserving causality) [6], [10], [11].

In Euclidean space, where $x^0 \to -ix_4$ (euclidean four-vector), and the field $\phi(x)$ changes to the real-valued field $\phi(\mathbf{x}, x_4)$ the action becomes:

$$S_E[\phi] = \frac{1}{2}\int d^4x\phi(x)(-\Box + m^2)\phi(x), \tag{2.3}$$

where $\Box$ is now the four-dimensional Laplacian

$$\Box = \sum_{\mu=1}^{4} \partial_\mu \partial_\mu.$$

By applying the euclidean action (2.3) in equation (1.10), we can derive the Green function for a field that satisfies the free Klein-Gordon equation in euclidean space-time [11]:

$$\langle \phi(x)\phi(y)\rangle = \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2+m^2}. \tag{2.4}$$

2.2 Dirac Equation for free fermionic fields

We denote the Dirac equation for free fermionic fields in Minkowski space:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0, \tag{2.5}$$

where m is the mass of a particle with spin 1/2, γ^μ are 4×4 Dirac matrices that follow the anticommuting relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.6}$$

and ψ is a four-component field, whose components are labeled by Greek indices (α, β, etc.).

The Dirac equation is derived using Hamilton's principle from the action dependent on the fields ψ and $\bar{\psi}$ ($\equiv \psi^\dagger\gamma^0$):

$$S_F[\psi, \bar{\psi}] = \int d^4x\bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x). \tag{2.7}$$

that arises from the Lagrangian density

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x).$$

Using the Lagrangian density and the Euler-Lagrange equations, we can also find the equation of motion for the conjugate field $\bar{\psi}$:

$$-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0. \tag{2.8}$$

The Dirac equation is a partial differential equation, that tries to solve the problems that come up in the solutions of the Klein Gordon equation. It is a first-order equation in respect to both time and space, as relativity dictates (time and space are treated equivalently in relativity), and only one initial condition is now needed. Also, negative probability densities no longer appear, but infinite negative energies do, and therefore the equation was rejected. Now, we use the Dirac equation as a multiparticle theory with the explanation of the Dirac Sea. In this model, all the states with negative energies are filled, according to Pauli's exclusion principle, and the vacancy of a negative-energy state can now be interpreted as the presence of an anti-particle (positron), and in this way, the existence of antimatter is introduced [6], [11].

As before, we can move to Euclidean space, by changing to imaginary times and replacing $iS_F[\psi, \bar{\psi}] \to -S_F^{(eucl.)}[\psi, \bar{\psi}]$, where $-S_F^{(eucl.)}[\psi, \bar{\psi}]$ is the euclidean action. In euclidean space, a four-dimensional rotation group replaces the Lorentz group and for convenience we use a new set of γ-matrices, γ_μ^E $(\mu = 1, 2, 3, 4)$, that satisfy the relation:

$$\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu}$$

We can choose the hermitian matrices $\gamma_4^E = \gamma^0$ and $\gamma_j^E = -i\gamma^j$, and the euclidean action can be written as:

$$S_F^{(eucl.)} = \int d^4x \bar{\psi}(x)(\gamma_\mu^E \partial_\mu + m)\psi(x) \tag{2.9}$$

By applying the euclidean action (2.9) in equation (1.11), we can derive the Green function for a field that satisfies the Dirac equation in euclidean space-time [11]:

$$\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle = \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} \frac{[-i\gamma_\mu p_\mu + m]_{\alpha\beta}}{p^2 + m^2} e^{ip\cdot(x-y)}. \tag{2.10}$$

2.3 QED Action and the Photon Propagator

In this section, we want to write down the gauge-invariant action that represents Quantum Electrodynamics (QED), that is, the theory that describes the electromagnetic interactions between light (photons) and matter (charged particles). We will derive this result, starting with the action of the free Dirac field, in which we will

gauge the global symmetry and then we will add a kinetic term for the gauge fields, in an analogous way to that presented by Rothe (2005) [11].

The free fermionic action in Minkowski space is:

$$S_F^{(0)} = \int \mathrm{d}^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \tag{2.11}$$

The following transformation leaves this action invariant

$$\psi(x) \to G\psi(x) \tag{2.12}$$
$$\bar{\psi}(x) \to \bar{\psi}(x)G^{-1} \tag{2.13}$$

where G is an element of the abelian $U(1)$ group, for instance

$$G = e^{i\Lambda} \tag{2.14}$$

and Λ is a global transformation (that is, independent of x). We also want the action to be invariant under local $U(1)$ transformations, i.e. when Λ is a function of space-time. The ordinary derivative term in eq. (2.11) is not invariant in this case. We can correct this by introducing the covariant derivative $\mathcal{D}_\mu$, which will replace the ordinary four-derivative ∂_μ and includes a four-vector potential $A_\mu(x)$. This covariant derivative is defined as:

$$\mathcal{D}_\mu = \partial_\mu + ieA_\mu. \tag{2.15}$$

The action then becomes:

$$S_F = \int \mathrm{d}^4x \bar{\psi}(x)(i\gamma^\mu \mathcal{D}_\mu - m)\psi(x) \tag{2.16}$$

and is invariant under the following local transformations:

$$\begin{aligned} &\psi(x) \to G(x)\psi(x) \\ &\bar{\psi}(x) \to \bar{\psi}(x)G^{-1}(x) \\ &A_\mu(x) \to G(x)A_\mu(x)G^{-1}(x) - \frac{i}{e}G(x)\partial_\mu G^{-1}(x) \\ &\text{where} \quad G(x) = e^{i\Lambda(x)} \end{aligned} \tag{2.17}$$

In this case, the transformation of the four-vector can be written as $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \Lambda$, which is familiar from classical electromagnetism. Also, since this is an abelian case,

A_μ and G commute and thus $GA_\mu G^{-1} = A_\mu$. Even though A_μ is transformed inhomogeneously, the invariance of the action (2.16) is ensured because of the homogeneous transformation of the covariant derivative:

$$\mathcal{D}_\mu \to G\mathcal{D}_\mu G^{-1} \tag{2.18}$$

It is necessary, besides the term added with the introduction of A_μ to ensure gauge invariance, to introduce a kinetic term that enables the propagation of the four-vector A_μ. This term should also leave the action invariant under local transformations, and takes the form

$$S_G = -\frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu}, \tag{2.19}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the gauge invariant field tensor (the tensor of electromagnetic fields). The total gauge-invariant action that describes the theory of Quantum Electrodynamics, that is, the dynamics of the fields $\psi, \bar{\psi}$ and A_μ in Minkowski space is:

$$S_{QED} = S_G + S_F = -\frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu} + \int \mathrm{d}^4x \bar{\psi}(x)(i\gamma^\mu\mathcal{D}_\mu - m)\psi(x) \tag{2.20}$$

The Euler-Lagrange equations that follow from the above action are:

$$\begin{aligned} &(i\gamma_\mu\mathcal{D}_\mu - m)\psi(x) = 0 \\ &\bar{\psi}(x)(-i\gamma^\mu\mathcal{D}_\mu - m) = 0 \\ &\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi \end{aligned} \tag{2.21}$$

The first two equations are Dirac equations for fermions (or anti-fermions) in an electromagnetic field, while the third one is a non-homogeneous Maxwell's equation with the "source" j^ν equal to $e\bar{\psi}\gamma^\nu\psi$, that is the fermionic current density.

Now, we transition into euclidean space. This requires real time to become imaginary ($x^0 \to -ix_4$) and thus the time derivative to change to $\partial_0 \to +i\partial_4$. Since we need the condition $A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\Lambda(x)$ to hold, even in euclidean space, the time-component of the four-potential becomes $A_0 \to +iA_4$. By substituting formally, equation (2.19) becomes:

$$S_G \to \frac{i}{4}\int \mathrm{d}^4x F_{\mu\nu}F_{\mu\nu}, \tag{2.22}$$

and a sum over μ and ν ($\mu, \nu = 1, 2, 3, 4$) is implied. Now, $\exp(iS_G)$ goes to $\exp(-S_G^{(eucl.)})$, which is an exponentially damping functional of A_μ. With the same substitutions (that is, transitioning to imaginary time), the form of the corresponding covariant derivative remains unchanged ($\mathcal{D}_\mu = \partial_\mu + ieA_\mu$), since both the partial time-derivative and the time-component of the four-potential changed in the same way. Thus, the action (2.20) becomes $iS^{(eucl.)}$, and specifically,

$$S_{QED}^{(eucl.)} = S_G^{(eucl.)} + S_F^{(eucl.)} = \frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F_{\mu\nu} + \int \mathrm{d}^4x \bar{\psi}(\gamma_\mu \mathcal{D}_\mu + m)\psi. \tag{2.23}$$

where the γ_μ ($\mu = 1, ..., 4$) matrices are the euclidean γ matrices that follow the relation $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

The Photon Propagator

We consider the functional integral, part of the photonic propagator in Minkowski space:

$$\int DA\ e^{iS_G[A]}, \tag{2.24}$$

where $S_G[A]$ is the gauge action for the electromagnetic field, and the measure of integration is $DA \equiv DA^0DA^1DA^2DA^3$. We can rewrite the action (after integrating by parts and expanding as a Fourier integral) as:

$$\begin{aligned} S_G &= -\frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}\int \mathrm{d}^4x A_\mu(x)(\Box g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x) \\ &= \frac{1}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\tilde{A}_\mu(k)(-k^2g^{\mu\nu} + k^\mu k^\nu)\tilde{A}_\nu(-k) \end{aligned} \tag{2.25}$$

where $\Box$ is the d'Alembertian and $g^{\mu\nu}$ is the inverse Minkowski metric. This expression equals zero when $\tilde{A}_\mu(k) = k_\mu\Lambda(k)$, for any scalar function $\Lambda(k)$. Then, the corresponding integrand of (2.24) equals one, and the functional integral over this set of field configurations diverges. A related problem appears in the equation

$$\begin{aligned} &(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)D_{ph}^{\nu\rho}(x-y) = i\delta^{(4)}(x-y) \\ &\text{or } (-k^2 g_{\mu\nu} + k_\mu k_\nu)\tilde{D}_{ph}^{\nu\rho}(k) = i\delta_\mu{}^\rho \end{aligned} \tag{2.26}$$

which defines $D_{ph}^{\nu\rho}$ as the photon propagator (the second expression is its Fourier transform): This equation forbids the existence of solutions for the photon propagator because the 4×4 matrix $(-k^2 g_{\mu\nu} + k_\mu k_\nu)$ is non-invertible.

This problem comes up because of gauge invariance. The action for the electromagnetic field is invariant under the transformation

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \Lambda(x)$$

The four-potential $A_\mu(x)$ vanishes, and hence is the cause of the problem, if its gauge equivalent $A_\mu(x)$ takes the form $\frac{1}{e}\partial_\mu\Lambda(x)$. The functional integral in this case, is not well defined, since we repeatedly integrate over infinite equivalent configurations. To fix the problem, we would like to count only one of the equivalent configurations, which is the part of the integral that interests us.

The solution was found, using a trick invented by Faddeev and Popov. Let $G(A)$ be a function that will vanish as we choose a gauge-fixing condition (i.e. $G(A) = \partial_\mu A^\mu$ is the Lorentz gauge). We can choose only the configurations with $G(A)$ to stay in the functional integral by inserting a functional delta function, $\delta(G(A))$ (which is an infinite product of delta functions, one for each space-time point x). To be able to do this properly, we insert unity in the functional integral, in the following form

$$1 = \int D\Lambda(x)\delta(G(A'))\det\left(\frac{\delta G(A')}{\delta\Lambda}\right), \tag{2.27}$$

where A' denotes the gauge-transformed field,

$$A'_\mu(x) = A_\mu(x) - \frac{1}{e}\partial_\mu\Lambda(x), \tag{2.28}$$

and eq. (2.27) is the continuum generalization of the identity

$$1 = \left(\prod_i \mathrm{d}a_i\right)\delta^{(n)}(\mathbf{g}(\mathbf{a}))\det\left(\frac{\partial g_i}{\partial a_j}\right) \tag{2.29}$$

for n-dimensional discrete vectors. For example, in Lorentz gauge, the function $G(A') = \partial^\mu A'_\mu = \partial^\mu A_\mu - (1/e)\Box\Lambda$ gives a functional determinant that is independent of A, $\det(\delta G(A')/\delta\Lambda) = \det(-\Box/e)$. Hence, we can treat the determinant as a constant and take it out of the functional integral. Then, the functional integral (2.24) takes the form

$$\det\left(\frac{\delta G(A')}{\delta\Lambda}\right)\int D\Lambda\int DA\; e^{iS_G[A]}\delta(G(A')). \tag{2.30}$$

The next step is to change variables, $A \to A'$. In this case, $DA' = DA$, and due to gauge invariance, $S_G[A] = S_G[A']$. Now, A' is a dummy variable and we can rename it back to A, and we get

$$\int DA\; e^{iS_G[A]} = \det\left(\frac{\delta G(A')}{\delta\Lambda}\right)\int D\Lambda \int DA\; e^{iS_G[A]}\delta(G(A)) \tag{2.31}$$

The delta function allows in the functional integral over A, only one of the physically equivalent gauge configurations, just like we wanted. The integral over Λ diverges and gives an infinite but constant multiplication factor.

To continue, we need to define a gauge fixing function $G(A)$. We choose the family of functions

$$G(A) = \partial^\mu A_\mu(x) - \omega(x), \tag{2.32}$$

where $\omega(x)$ is any scalar function. When we set this $G(A)$ to zero, we have a generalized version of the Lorentz condition. Since the scalar function $\omega(x)$ is independent of A, the functional determinant is the same as in the Lorentz gauge, $\det(\delta G(A')/\delta\Lambda) = \det(-\Box/e)$. Consequently, the functional integral takes the form

$$\int DA\; e^{iS_G[A]} = \det\left(-\frac{1}{e}\Box\right)\left(\int D\Lambda\right)\int DA\; e^{iS_G[A]}\delta(\partial^\mu A_\mu - \omega(x)) \tag{2.33}$$

This expression is true for any scalar function $\omega(x)$, and therefore will be true even if we replace the right-hand side with any properly normalized linear combination of different functions $\omega(x)$. As a final trick, we will integrate over all $\omega(x)$, with a Gaussian weighted function, centered at $x = 0$. The above expression is now given by

$$\begin{aligned} &N(\alpha)\int D\omega\, \exp\left[-i\int \mathrm{d}^4x\frac{\omega^2}{2\alpha}\right]\det\left(-\frac{1}{e}\Box\right)\left(\int D\Lambda\right)\int DA\; e^{iS_G[A]}\delta(\partial^\mu A_\mu - \omega(x)) \\ &\quad = N(\alpha)\det\left(-\frac{1}{e}\Box\right)\left(\int D\Lambda\right)\int DA\; e^{iS_G[A]}\exp\left[-i\int \mathrm{d}^4x\frac{1}{2\alpha}(\partial^\mu A_\mu)^2\right], \end{aligned} \tag{2.34}$$

where $N(\alpha)$ is the normalization factor and the delta function has been used to calculate the integral over $\omega(x)$. The parameter α can be chosen freely. Effectively, a term $-(\partial^\mu A_\mu)^2/2\alpha$ is added to the Lagrangian.

For any Green function of any gauge-invariant operator $\mathcal{O}(A)$

$$\langle\Omega|\, T\mathcal{O}(A)\,|\Omega\rangle = \frac{\int DA\ \mathcal{O}(A)\ \exp\left[i\int \mathrm{d}^4x\mathcal{L}\right]}{\int DA\ \exp\left[i\int \mathrm{d}^4x\mathcal{L}\right]}, \tag{2.35}$$

the multiplicative factors that go to infinity cancel out, as the same manipulations can be performed both on the numerator and the denominator (the process above calculated the functional integral on the denominator). If the operator $\mathcal{O}(A)$ is not gauge-invariant, we cannot change variables from A to A', and the rest of the process would not be valid. The resulting Green function is

$$\langle\Omega|\, T\mathcal{O}(A)\,|\Omega\rangle = \frac{\int DA\ \mathcal{O}(A)\ \exp\left[i\int \mathrm{d}^4x\left[\mathcal{L}-\frac{1}{2\alpha}(\partial^\mu A_\mu)^2\right]\right]}{\int DA\ \exp\left[i\int \mathrm{d}^4x\left[\mathcal{L}-\frac{1}{2\alpha}(\partial^\mu A_\mu)^2\right]\right]}. \tag{2.36}$$

We see that the only effect of the Faddeev-Popov procedure is the extra gauge fixing term added to the action.

The new equation that gives the photon propagator, after the addition of this gauge-fixing term, is

$$\left(-k^2 g_{\mu\nu} + \left(1-\frac{1}{\alpha}\right)k_\mu k_\nu\right)\tilde{D}^{\nu\rho}_{ph}(k) = i\delta_\mu{}^\rho,$$

which has the solution

$$\tilde{D}^{\mu\nu}_{ph}(k) = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\alpha)\frac{k^\mu k^\nu}{k^2}\right). \tag{2.37}$$

This is the expression we wanted to find for the photon propagator. The $i\epsilon$ term arises in order to cancel any infinities that appear when $k^2 = (k^0)^2 - (k^1)^2 - (k^2)^2 - (k^3)^2 = 0$.

In euclidean space, the photon propagator takes the form

$$\tilde{D}^{ph,E}_{\mu\nu}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) \tag{2.38}$$

For practical purposes, we choose a specific value for α to be able to make any computations. For convenience we either choose

$$\alpha = 0,\ \text{Landau gauge;}$$
$$\text{or}$$
$$\alpha = 1,\ \text{Feynman gauge.}$$

It is guaranteed by the Faddeev-Popov procedure that the value of any Green function of any gauge-invariant operator computed from Feynman diagrams will not

depend on the value of α that was chosen for the computations, as long as it was the same consistently.

In this section, the Faddeev-Popov procedure and the derivation of the photon propagator were presented in an equivalent way to that explained by Peskin and Schroeder (1995) [10].

2.4 QCD Action and the Gluon Propagator

In this section, we want to derive the gauge-invariant action that describes Quantum Chromodynamics (QCD), that is the theory that studies the strong interactions, between gluons and "colored" particles (quarks). Quarks are fundamental particles that make up hadrons, such as the proton and the neutron and mesons, usch as the pion, and they come in six different flavors, with different mass: up, down, charm, strange, top and bottom. They are "colored", because they can come in three different color charges: red, blue and green. As before, the final expression for the action must be derived by combining the action of the free gluonic field, which provides the kinetic term for gauge fields and the Dirac action for fermions that interact with the gluonic field. The derivation of the continuum QCD action is presented in an analogous way to that described by Rothe (2005) [11].

The action we are looking for is an extension of the abelian case we studied before, that replaces the group $U(1)$, with a non-abelian unitary group. So, we consider that we have N free Dirac fields $\psi^a,\ a = (1, ..., N)$ instead of a single one, of mass m_0. We now introduce the N-component column and row vectors

$$\underset{\sim}{\psi} = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix}, \quad \underset{\sim}{\bar{\psi}} = \left(\bar{\psi}^1, \ldots, \bar{\psi}^N\right). \tag{2.39}$$

Then, the free Dirac action, in Minkowski space, takes the form

$$S_F^{(0)} = \int \mathrm{d}^4x \underset{\sim}{\bar{\psi}}(x)(i\gamma^\mu\partial_\mu - m_0)\underset{\sim}{\psi}(x). \tag{2.40}$$

while the interaction field (in this case, the gluonic field), has to become a $N \times N$ matrix $\underset{\sim}{A}_\mu$. The fermionic action will then be:

$$S_F = \int \mathrm{d}^4x \underset{\sim}{\bar{\psi}}(x)(i\gamma^\mu\mathcal{D}_\mu - m_0)\underset{\sim}{\psi}(x) \tag{2.41}$$

where $\mathcal{D}_\mu = \partial_\mu + ig_0\underset{\sim}{A}_\mu$ is the covariant derivative and g_0 is the bare coupling constant of the theory.

The action is invariant under local unitary transformations in N dimensions, and in particular under the non-abelian subgroup $SU(N)$:

$$\begin{aligned}
&\underset{\sim}{\psi}(x) \to \underset{\sim}{G}(x)\underset{\sim}{\psi}(x) \\
&\underset{\sim}{\bar{\psi}}(x) \to \underset{\sim}{\bar{\psi}}(x)\underset{\sim}{G}^{-1}(x) \\
&\underset{\sim}{A}_\mu(x) \to \underset{\sim}{G}(x)\underset{\sim}{A}_\mu(x)\underset{\sim}{G}^{-1}(x) - \frac{i}{g_0}\underset{\sim}{G}(x)\partial_\mu \underset{\sim}{G}^{-1}(x)
\end{aligned} \tag{2.42}$$

where $\underset{\sim}{G}(x)$ is an element of the group $SU(N)$, and can be written as $\underset{\sim}{G}(x) = e^{i\Lambda(x)}$, where $\Lambda(x)$ is a hermitian traceless matrix of the Lie algebra of $SU(N)$.

Since quarks can come in three colored charges, and hadrons have been observed to be color singlets with respect to the group $SU(3)$, we expect that the theory follows the Lie algebra of this group and we will use that $N = 3$.

The next step is to add to the action a kinetic term for gluons. The tensor that describes strong interactions must be covariant under local transformations of the subgroup $SU(N)$, that is, $\underset{\sim}{F}_{\mu\nu} \to \underset{\sim}{G}(x)\underset{\sim}{F}_{\mu\nu}\underset{\sim}{G}^{-1}(x)$. Also, since the gluonic field $\underset{\sim}{A}_\mu$ is a 3×3 matrix, the definition of $\underset{\sim}{F}_{\mu\nu}$ must change, compared to the abelian case, to be a covariant tensor. We write down the matrix-valued field tensor in continuum QCD:

$$\underset{\sim}{F}_{\mu\nu} = \partial_\mu \underset{\sim}{A}_\nu - \partial_\nu \underset{\sim}{A}_\mu + ig_0[\underset{\sim}{A}_\mu, \underset{\sim}{A}_\nu] \tag{2.43}$$

The most simple term that we can add to the action, that is gauge-invariant, must be proportional to the trace of the quantity $\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu}$.

Different types of gluons can interact with each other, and the gluonic part of the action has both a kinetic and an interaction term. This result, in Minkowski space, takes the form

$$S_G = -\frac{1}{2}\int \mathrm{d}^4x \,\mathrm{Tr}\big(\underset{\sim}{F}_{\mu\nu}\underset{\sim}{F}^{\mu\nu}\big) \tag{2.44}$$

$\underset{\sim}{A}_\mu$ is a hermitian and traceless matrix, and therefore is an element of the Lie algebra of $SU(3)$ and can be written as

$$\underset{\sim}{A}_\mu(x) = \sum_{a=1}^{8} A^a_\mu(x)\frac{\lambda^a}{2} \tag{2.45}$$

where $A^a_\mu(x)$ are eight real-valued vector fields, corresponding to the eight generators of $SU(3)$. The eight group generators λ^a can be chosen to be the (3×3) Gell-Mann matrices, that satisfy the relations

$$\begin{aligned}
[\lambda^a, \lambda^b] &= 2i\sum_{c=1}^{8} f^{abc}\lambda^c, \\
\mathrm{Tr}\big(\lambda^b\lambda^c\big) &= 2\delta_{bc}
\end{aligned} \tag{2.46}$$

where f^{abc} are the completely antisymmetric structure constants of the group. We can also write down the tensor $\underset{\sim}{F}_{\mu\nu}$ as an element of Lie algebra of $SU(3)$ as:

$$\underset{\sim}{F}_{\mu\nu}(x) = \sum_{a=1}^{8} F^a_{\mu\nu} \frac{\lambda^a}{2} \tag{2.47}$$

Since the generators of the group $SU(3)$ satisfy the relations (2.46), we can find the expression that connects the eight components of $F^a_{\mu\nu}$ and A^a_μ:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_0 f_{abc} A^b_\mu A^c_\nu \tag{2.48}$$

Now the gluonic part of the action also be written as:

$$S_G = -\frac{1}{4} \int \mathrm{d}^4x F^a_{\mu\nu} F^{a\mu\nu} \tag{2.49}$$

The total gauge-invariant action under local $SU(3)$ transformations, that describes the theory of Quantum Chromodynamics, that is, the dynamics of the fields $\underset{\sim}{\psi}, \underset{\sim}{\bar{\psi}}$ and $\underset{\sim}{A}_\mu$ in Minkowski space is:

$$S_{QCD} = S_G + S_F = -\frac{1}{4} \int \mathrm{d}^4x F^a_{\mu\nu} F^{a\mu\nu} + \int \mathrm{d}^4x \underset{\sim}{\bar{\psi}}(x)(i\gamma^\mu \mathcal{D}_\mu - m_0)\underset{\sim}{\psi}(x) \tag{2.50}$$

and in euclidean space, where $\exp(iS)$ goes to $\exp(-S^{(eucl.)})$, the action becomes

$$S^{(eucl.)}_{QCD} = S^{(eucl.)}_G + S^{(eucl.)}_F = \frac{1}{4} \int \mathrm{d}^4x F^a_{\mu\nu} F^a_{\mu\nu} + \int \mathrm{d}^4x \underset{\sim}{\bar{\psi}}(x)(\gamma_\mu \mathcal{D}_\mu + m_0)\underset{\sim}{\psi}(x). \tag{2.51}$$

The Gluon Propagator

We consider the functional integral, part of the gluonic propagator, in Minkowski space:

$$\int DA \, \exp\left[i \int \mathrm{d}^4x \left(-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \right) \right] \tag{2.52}$$

where $DA = \prod_x \prod_{\mu=0}^{3} \prod_{a=1}^{8} dA^a_\mu$. But, as in the abelian case, this functional integral diverges, because of the infinite field configurations corresponding to local gauge transformations. As before, we will follow the Faddeev-Popov procedure (as described by Peskin and Schroeder, 1995 [10]) to restrict the field configurations, by applying

a gauge-fixing condition $G(A) = 0$ at each space-time point x. We insert unity into the functional integral in the following way:

$$1 = \int D\Lambda(x)\delta(G(A'))\det\left(\frac{\delta G(A')}{\delta\Lambda}\right), \tag{2.53}$$

where A' is the gauge field A transformed through a infinitesimal finite gauge transformation

$$(A')^a_\mu = A^a_\mu + \frac{1}{g_0}\partial_\mu\Lambda^a + f^{abc}A^b_\mu\Lambda^c. \tag{2.54}$$

Since the action is gauge-invariant, A can be replaced by A', and we can change the order of functional integration between A and Λ. Then, we can also change variables in the A integral, from A to A'. This transformation, is a linear shift of the A^a_μ, followed by a unitary rotation of the components of the symmetry multiplet $A^a_\mu(x)$ at each point, and the integration measure is preserved by both of these operations. Therefore, $DA = DA'$ under the integral over Λ. As in the abelian case, we can factor out the integral over gauge motions Λ into an overall normalization, and we are left with

$$\int DA\; e^{iS_G[A]} = \left(\int D\Lambda\right)\int DA\; e^{iS_G[A]}\delta(G(A))\det\left(\frac{\delta G(A')}{\delta\Lambda}\right). \tag{2.55}$$

When calculating Green functions of gauge-invariant operators, this normalization factor cancels out, as in the abelian case.

The procedure to derive the gluonic propagator is the same as in the case of the photonic propagator (presented by Peskin and Schroeder, 1995 [10]). We choose the generalized Lorentz gauge condition

$$G(A) = \partial^\mu A^a_\mu(x) - \omega^a(x), \tag{2.56}$$

with a Gaussian weight for ω^a, and with the same manipulations we find gluon propagator

$$D^{ab}_{\mu\nu,G}(x-y) = \langle A^a_\mu(x)A^b_\nu(y)\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}e^{-ik\cdot(x-y)} \tag{2.57}$$

and in momentum space

$$\tilde{D}^{ab}_{\mu\nu,G}(k) = \frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}, \tag{2.58}$$

where the gauge parameter α can be chosen freely.

While in the abelian case the determinant is independent of A and could be treated as a contribution to the normalization factor, in the non-abelian case, this holds no more. We can evaluate

$$\frac{\delta G(A')}{\delta \Lambda} = \frac{1}{g_0}\partial^\mu \mathcal{D}_\mu, \tag{2.59}$$

which is an operator that depends on A. In this case, the functional determinant contributes extra terms to the action.

Faddeev and Popov represented this determinant as a functional integral over Grassmann fields:

$$\det\left(\frac{1}{g_0}\partial^\mu \mathcal{D}_\mu\right) = \int DcD\bar{c}\, \exp\left[i\int \mathrm{d}^4x \bar{c}(-\partial^\mu \mathcal{D}_\mu)c\right], \tag{2.60}$$

where the factor $1/g_0$ is absorbed into the normalization of the fields c and $\bar{c}$. These fields, c and $\bar{c}$ and their particle excitations are named Faddeev-Popov ghosts. The Lagrangian that describes ghost fields follows from equation (2.60) and can be written as

$$\mathcal{L}_{\text{ghost}} = \bar{c}^a(-\Box\delta^{ac} - g_0\partial^\mu f^{abc}A^b_\mu)c^c. \tag{2.61}$$

We can find the ghost propagator from the first term

$$\langle c^a(x)\bar{c}^b(y)\rangle = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{i}{k^2}\delta^{ab}e^{-ik\cdot(x-y)} \tag{2.62}$$

and in momentum space

$$\tilde{D}_{\text{ghost}}(k) = \frac{i}{k^2}\delta^{ab}. \tag{2.63}$$

The final action of QCD, including all of the effects of Faddeev-Popov gauge fixing, in Minkowski space is

$$S^{total}_{QCD} = \int \mathrm{d}^4x(-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \underset{\sim}{\bar{\psi}}(x)(i\gamma^\mu\mathcal{D}_\mu - m_0)\underset{\sim}{\psi}(x) - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 + \bar{c}^a(-\partial^\mu\mathcal{D}^{ac}_\mu)c^c). \tag{2.64}$$

Chapter 3

Quantum Field Theory on the Lattice

3.1 The Lattice as a regulator of a QFT

As we have seen in previous sections, in order for a path integral expression to be well defined, the system described by the corresponding Green functions, needs to have a denumerable number of degrees of freedom. Since the fields are defined at every space-time point, the degrees of freedom in such systems are infinite, and thus the functional integrals are only formally defined. Now, to be able to give a precise definition to these integrals, we need to discretize both time and space, that is, we need to introduce a space-time lattice. In this case, the lattice spacing a, which would be the distance between two neighboring lattice points, is the regulator of the theory. We expect that after we finish with all the calculations we want, we will be able to return to a continuum space-time, by taking the limit $a \to 0$.

Specifically, the computation of Green functions through lattice regularization is divided into two main steps. The first step is to introduce the space-time lattice, with lattice spacing a, into the Green function. Then, the functional integrals have a denumerable number of degrees of freedom, just like we wanted, and the computation of the Green functions becomes easy. These results, though, depend on the lattice spacing a, and when, in the second step, we want to find the continuum limit (that is take the limit $a \to 0$), some unwanted infinities may appear. So, to address these unwanted infinities, we need to renormalize the theory, and tune any bare parameters to the lattice spacing, in a specific way that depends on the dynamics of the system, and leaves any observable physical quantities insensitive to the underlying existence of the lattice.

Unfortunately, weak coupling perturbation theory on the lattice is difficult for

two reasons. Firstly, the integrands of the functional path integrals have unusual structure and are modified in a non-trivial way. Secondly, in the lattice formulation of gauge theories, new interaction vertices appear, that do not appear in the continuum limit [6], [11].

It is more convenient to compute Green functcions in momentum space, instead of in coordinate space. Therefore, we introduce the Fourier representation of a function $f(x)$ of a single continuous variable [11]. If its absolute value is square integrable, we have

$$f(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \tilde{f}(k) e^{ikx}. \tag{3.1}$$

However, if x is restricted only on lattice points (that is, multiples of the lattice spacing a), where $x = na,\ n \in \mathbb{Z}$, then the Fourier representation of $f(na)$ is

$$f(na) = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} \tilde{f}_a(k) e^{ikna}, \tag{3.2}$$

where $\tilde{f}_a(-\pi/a) = \tilde{f}_a(\pi/a)$. Now, the values of momentum are restricted in the Brillouin zone (BZ), $[-\pi/a, \pi/a]$. We can also represent $\tilde{f}_a(k)$ by a Fourier series:

$$\tilde{f}_a(k) = a \sum_{n=-\infty}^{\infty} f(na) e^{-inka}. \tag{3.3}$$

We can derive this equation by discretizing the inverse Fourier transformation of the continuous function $f(x)$. We can also get a Fourier representation for the periodic $\delta-$function in the Brillouin zone, (δ_P), just by setting $f(na) = 1/2\pi$ in the previous equation.

$$\delta_P(k) = \frac{a}{2\pi} \sum_n e^{-inka}. \tag{3.4}$$

This function has non-vanishing support at $k = 0 \mod 2n\pi$. The equivalent of the Dirac $\delta-$function, $\delta(x-y)$, on the lattice is the Kronecker-δ (multiplied by $1/a$):

$$\delta_{nm} = a \int_{-\pi/a}^{\pi/a} \frac{dp}{2\pi} e^{ip(n-m)a}. \tag{3.5}$$

All these equations can be trivially extended to multivariable functions, and specifically, in a four-dimensional space-time, where the number of variables would be four.

3.2 The Free Scalar Field on the lattice

We have derived in the previous chapter the action of a free Klein-Gordon field in euclidean space:

$$S_E[\phi] = \frac{1}{2}\int d^4x\phi(x)(-\Box + M^2)\phi(x), \tag{3.6}$$

where M is the mass of the scalar field and $\Box$ is the four-dimensional Laplacian

$$\Box = \sum_{\mu=1}^{4} \partial_\mu \partial_\mu. \tag{3.7}$$

Also, in imaginary time formulation, we know that the Green function becomes correlation functions of a statistical mechanical system defined by the partition function

$$Z = \int D\phi e^{-S[\phi]}, \tag{3.8}$$

where the formal definition of $D\phi$ is

$$D\phi = \prod_{\mathbf{x},x_4} \mathrm{d}\phi(\mathbf{x}, x_4). \tag{3.9}$$

Now, we introduce the space-time lattice with lattice spacing a. Every point on the lattice is denoted by four integers, and specifically, $n \equiv (n_1, n_2, n_3, n_4)$, in which the last integer denotes euclidean time. We transition from the continuum formulation to the lattice by applying the following substitutions:

$$\begin{aligned}
&x_\mu \to n_\mu a,\\
&\phi(x) \to \phi(na),\\
&\int \mathrm{d}^4x \to a^4 \sum_n,\\
&\Box\phi(x) \to \frac{1}{a^2}\hat{\Box}\phi(na),\\
&D\phi \to \prod_n d\phi(na),
\end{aligned} \tag{3.10}$$

where the definition of the action of the dimensionless lattice Laplacian $\hat{\Box}$ is

$$\hat{\Box}\phi(na) = \sum_\mu (\phi(na + \hat{\mu}a) + \phi(na - \hat{\mu}a) - 2\phi(na)), \tag{3.11}$$

and $\hat{\mu} \equiv \hat{e}_\mu$ is a unit vector pointing along the μ-direction. The discretization of the free scalar field is presented in this section in a similar way to that described by Rothe (2005) [11].

Next, we would like to derive a path integral expression that contains only dimensionless variables. So, we scale the mass parameter M and the field ϕ, which both have dimension of inverse length, according to their "canonical" dimension. The definitions of the dimensionless quantities $\hat{M}$ and $\hat{\phi}_n$ are

$$\begin{aligned} \hat{\phi}_n &= a\phi(na), \\ \hat{M} &= aM. \end{aligned} \tag{3.12}$$

Then, the dimensionless correlation function can be written as

$$\langle \hat{\phi}_n \hat{\phi}_m \dots \rangle = \frac{\int \prod_l \mathrm{d}\hat{\phi}_l \ \hat{\phi}_n \hat{\phi}_m \dots e^{-S_E[\hat{\phi}]}}{\int \prod_l \mathrm{d}\hat{\phi}_l \ e^{-S_E[\hat{\phi}]}}, \tag{3.13}$$

where

$$S_E = -\frac{1}{2} \sum_{n,\hat{\mu}} \hat{\phi}_n \hat{\phi}_{n+\mu} + \frac{1}{2}(8 + \hat{M}^2) \sum_n \hat{\phi}_n \hat{\phi}_n, \tag{3.14}$$

and the sum over μ extends over all positive and negative directions.

This choice of lattice action is not unique; it is, however, the simplest one. Hence, the only requirement that any lattice action should fulfill is to reproduce the correct naive continuum limit. The scalar field is the only case in which the simplest lattice discretization gives the correct description of the quantum theory.

To calculate the correlation function, we write the action in the form

$$S_E = \frac{1}{2} \sum_{n,m} \hat{\phi}_n K_{nm} \hat{\phi}_m, \tag{3.15}$$

where K_{nm} is given by

$$K_{nm} = -\sum_{\mu>0} [\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{nm}] + \hat{M}^2 \delta_{nm}. \tag{3.16}$$

From Wick's theorem (see Appendix B), we know that an essential ingredient for the calculation of these correlation functions is the 2-point function, given by

$$\langle \hat{\phi}_n \hat{\phi}_m \rangle = (K^{-1})_{nm}, \tag{3.17}$$

where the inverse matrix K^{-1} is defined by the equation

$$\sum_l K_{nl}(K^{-1})_{lm} = \delta_{nm}, \tag{3.18}$$

and δ_{nm} is expressed in momentum space as

$$\delta_{nm} = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4\hat{k}}{(2\pi)^4} e^{i\hat{k}\cdot(n-m)}. \tag{3.19}$$

The hat is introduced on $\hat{k} = (\hat{k}_1, ..., \hat{k}_4)$, to emphasize on the fact that the variables are dimensionless. Using the Fourier representation, we can express K_{nm} as

$$K_{nm} = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4\hat{k}}{(2\pi)^4} \tilde{K}(\hat{k}) e^{i\hat{k}\cdot(n-m)}, \tag{3.20}$$

where

$$\tilde{K}(\hat{k}) = 4\sum_{\mu=1}^{4} \sin^2\frac{\hat{k}_\mu}{2} + \hat{M}^2. \tag{3.21}$$

The inverse matrix, and therefore the two-point Green function can be calculated easily from its definition and the ansatz

$$(K^{-1})_{nm} = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4\hat{k}}{(2\pi)^4} G(\hat{k}) e^{i\hat{k}\cdot(n-m)}. \tag{3.22}$$

Then, one uses the definition for the periodic delta function to arrive at the expression:

$$(K^{-1})_{nm} = \langle\hat{\phi}_n\hat{\phi}_m\rangle = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4\hat{k}}{(2\pi)^4} \frac{e^{i\hat{k}\cdot(n-m)}}{4\sum_\mu \sin^2\frac{\hat{k}_\mu}{2} + \hat{M}^2}. \tag{3.23}$$

This result depends on the lattice sites n, m and on the dimensionless mass parameter $\hat{M}$. To find the physical two point function, $\langle\phi(x)\phi(y)\rangle$, we define

$$\mathcal{G}(n, m; \hat{M}) \equiv \langle\hat{\phi}_n\hat{\phi}_m\rangle.$$

The next step would be to reintroduce the lattice spacing by rescaling $\hat{\phi}_n$ and $\hat{M}$ and take the limit $a \to 0$, while holding M, ϕ, $x = na$ and $y = ma$ fixed. In a more general case, in order to do that, we have to know which quantities to hold fixed

when removing the lattice structure. For instance, in an interacting theory, the mass parameter M is not a physical constant and should not be held fixed when taking the continuum limit. Nonetheless, in this case, this naive procedure gives the correct continuum limit. The two point function

$$\langle \hat{\phi}_n \hat{\phi}_m \rangle = \lim_{a \to 0} \frac{1}{a^2} \mathcal{G}(\frac{x}{a}, \frac{y}{a}; Ma) \tag{3.24}$$

approaches a finite limit (that is $\mathcal{G}(x/a, y/a; Ma)$ goes to zero) and gives the known result for the scalar two-point function, as was written in the previous chapter. By changing variables, we arrive at the following result

$$\mathcal{G}(\frac{x}{a}, \frac{y}{a}; Ma) = a^2 \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{\sum_\mu \tilde{k}_\mu^2 + M^2} \tag{3.25}$$

where $\tilde{k}_\mu$ is given by

$$\tilde{k}_\mu = \frac{2}{a} \sin \frac{k_\mu a}{2}. \tag{3.26}$$

Since the integration is restricted in the Brillouin zone $[-\frac{\pi}{a}, \frac{\pi}{a}]$, the momenta of the integral will be small compared to the inverse lattice spacing, and we can set $\tilde{k}_\mu \to k_\mu$. Then, we take the limit $a \to 0$, and we reproduce the known result:

$$\langle \phi(x)\phi(y) \rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 + M^2}. \tag{3.27}$$

The case of a free scalar field was a very simple one, because the transition to a space-time lattice was easy. The action chosen was the most simple one, with no extra terms that vanish when taking the limit $a \to 0$, and for this reason we call this discretization "naive". In addition, since the fields are free, and there are no interaction terms, the mass parameter M of the action is also the physical mass of the field and is held fixed as we let the lattice spacing go to zero [11].

3.3 Fermions on the lattice

The Doubling Problem

The euclidean Dirac action for fermions, as discussed before, is:

$$S_F^{(eucl.)} = \int d^4x \bar{\psi}(x)(\gamma_\mu^E \partial_\mu + M)\psi(x) \tag{3.28}$$

where M is the mass of the fermionic field.

The Green function we want to compute takes the form:

$$\langle \psi_\alpha(x)...\bar{\psi}_\beta(y)...\rangle = \frac{\int D\bar{\psi}D\psi(\psi_\alpha(x)...\bar{\psi}_\beta(y)...)e^{-S_F^{(eucl.)}[\psi,\bar{\psi}]}}{\int D\bar{\psi}D\psi e^{-S_F^{(eucl.)}[\psi,\bar{\psi}]}} \tag{3.29}$$

To find this fermionic Green function on the lattice, we will try to follow the same process as in the scalar field, so we will apply the naive discretization of space-time (as presented by Rothe, 2005 [11]). In the lattice formulation, the fields ψ and $\bar{\psi}$ must live only on the lattice sites na, where a is the lattice spacing, and the integration measure will be given by

$$D\bar{\psi}D\psi = \prod_{\alpha,n} \mathrm{d}\bar{\psi}_\alpha(na) \prod_{\beta,m} \mathrm{d}\psi_\beta(ma)$$

As before, we write the action in terms of dimensionless variables, by scaling M, ψ and $\bar{\psi}$ with a according to their canonical dimensions. This is attained by making the following substitutions

$$\begin{aligned} M &\to \frac{1}{a}\hat{M}, \\ \psi_\alpha(x) &\to \frac{1}{a^{3/2}}\hat{\psi}_\alpha(n), \\ \bar{\psi}_\alpha(x) &\to \frac{1}{a^{3/2}}\bar{\hat{\psi}}_\alpha(n), \\ \partial_\mu\psi_\alpha(x) &\to \frac{1}{a^{5/2}}\hat{\partial}_\mu\hat{\psi}_\alpha(n), \end{aligned} \tag{3.30}$$

where the antihermitian lattice derivative $\hat{\partial}_\mu$ is given by

$$\hat{\partial}_\mu\hat{\psi}_\alpha(n) = \frac{1}{2}[\hat{\psi}_\alpha(n+\hat{\mu}) - \hat{\psi}_\alpha(n-\hat{\mu})]. \tag{3.31}$$

Now, the free Dirac action takes the form

$$S_F^{(eucl.)} = \sum_{\substack{n,m \\ \alpha,\beta}} \bar{\hat{\psi}}_\alpha(n) K_{\alpha\beta}(n,m) \hat{\psi}_\beta(m), \tag{3.32}$$

where

$$K_{\alpha\beta}(n,m) = \sum_\mu \frac{1}{2}(\gamma_\mu)_{\alpha\beta}[\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}] + \hat{M}\delta_{mn}\delta_{\alpha\beta}. \tag{3.33}$$

With these substitutions, the dimensionless correlation functions can be calculated using the expression

$$\langle\hat{\psi}_\alpha(x)...\bar{\hat{\psi}}_\beta(y)...\rangle = \frac{\int D\bar{\hat{\psi}}D\hat{\psi}(\hat{\psi}_\alpha(x)...\bar{\hat{\psi}}_\beta(y)...)e^{-S_F^{(eucl.)}}}{\int D\bar{\hat{\psi}}D\hat{\psi}e^{-S_F^{(eucl.)}}} \tag{3.34}$$

and the integration measure is given by

$$D\bar{\hat{\psi}}D\hat{\psi} = \prod_{\alpha,n}\mathrm{d}\bar{\hat{\psi}}_\alpha(n)\prod_{\beta,m}\mathrm{d}\hat{\psi}_\beta(m) \tag{3.35}$$

From Wick's theorem (see Appendix B), we know that an essential ingredient for the calculation of these correlation functions is the 2-point function, given by

$$\langle\hat{\psi}_\alpha(n)\bar{\hat{\psi}}_\beta(m)\rangle = (K^{-1})_{\alpha\beta}(n,m). \tag{3.36}$$

As in the case of the free scalar field, Eq. (3.32) is the simplest action we can get, as it was a result of the most naive discretization. We want to check if even in the fermionic case, it gives the correct physical results. As before, we will compute the physical correlation function $\langle\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle$ by taking the continuum limit, as follows

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle = \lim_{a\to 0}\frac{1}{a^3}\mathcal{G}_{\alpha\beta}(\frac{x}{a},\frac{y}{a};Ma),$$

where $\mathcal{G}_{\alpha\beta}(n,m,\hat{M}) \equiv (K^{-1})_{\alpha\beta}(n,m)$, and the factor $1/a^3$ comes up when rescaling the fields. The definition of the inverse matrix $(K^{-1})_{\alpha\beta}(n,m)$ comes from the expression

$$\sum_{\lambda,l}(K^{-1})_{\alpha\lambda}(n,l)K_{\lambda\beta}(l,m) = \delta_{\alpha\beta}\delta_{nm},$$

and with the same calculations we did in the previous section, we arrive at the expression

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle = \lim_{a\to 0}\int_{-\pi/a}^{\pi/a}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{[-i\sum\gamma_\mu\tilde{\tilde{p}}_\mu + M]_{\alpha\beta}}{\sum_\mu\tilde{\tilde{p}}_\mu^2 + M^2}e^{ip\cdot(x-y)}, \tag{3.37}$$

where $\tilde{\tilde{p}}_\mu$ is given by

$$\tilde{\tilde{p}}_\mu = \frac{1}{a}\sin(p_\mu a). \tag{3.38}$$

In the limit $a \to 0$, $\tilde{\hat{p}}_\mu \to p_\mu$, and the integral reproduces the known 2-point function. Although this is very similar to the scalar case, an important difference exists. The argument of the sine-function in the scalar case is only half of that in the Dirac case. This is a large contrast between the two cases and the start of what we call the "fermion doubling" problem. Although in the case of the scalar field we can actually replace $\tilde{k}_\mu$ with k_μ when taking the continuum limit, the same is no longer true for the fermionic case. The sine-function in the correlation function vanishes at the edges of the Brillouin zone $[-\pi/a, \pi/a]$. Therefore, there are 16 $(= 2^4)$ regions of integration, where $\tilde{\hat{p}}_\mu$ takes a finite value in the limit $a \to 0$. High momentum excitations of the order of π/a (and $-\pi/a$) are involved in fifteen of these regions, and a momentum distribution function that resembles a single particle propagator arises. So, in the continuum limit, there are contributions to the 2-point Green function from sixteen fermion-like excitations in momentum space, of which fifteen are an artifact of the lattice and do not have a continuum equivalent.

In the continuum limit, for vanishing mass, the action is invariant under the global chiral transformation

$$\psi \to e^{i\theta\gamma_5}\psi \quad ; \quad \bar{\psi} \to \bar{\psi}e^{i\theta\gamma_5}, \tag{3.39}$$

where θ is a parameter, and $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ is a hermitian matrix that anticommutes with γ_μ $(\mu = 1, 2, 3, 4)$. Naively, by Noether's theorem, this means that there is a corresponding conserved current (the "axial" current). However, the existence of quantum fluctuations results in an anomalous divergence of this current, but in a lattice regularized theory, this symmetry implies the strict conservation of this current for any lattice spacing. The extra excitations, that is the doublers, that come up in the lattice formulation, cancel the anomaly of the continuum theory arising from momentum excitations around $\hat{p} = 0$ [11].

Wilson Fermions

To eliminate the fermion doubling problem, we need to choose a more complicated action than before, that also gives the correct continuum limit. To do that, we will add an extra term to the action that vanishes in the continuum limit. However, we are forced to break explicitly the chiral symmetry for vanishing fermion mass of the original theory; this formulation is described by Rothe (2005) [11]. We consider the action (in euclidean space, but with no label to remind us):

$$S_F^{(W)} = S_F - \frac{r}{2}\sum_n \bar{\hat{\psi}}(n)\hat{\Box}\hat{\psi}(n), \tag{3.40}$$

where r is the Wilson parameter and $\hat{\Box}$ is the 4-dimensional dimensionless Laplacian, as defined in the scalar case. If we rescale the parameters $\hat{\psi} = a^{3/2}\psi$ and $\hat{\Box} = a^2\Box$, it is obvious that the term added to the action vanishes linearly with a in the naive continuum limit. With similar substitutions as in the previous cases, we can write down the Wilson action as follows

$$S_F^{(W)} = \sum_{n,m} \bar{\hat{\psi}}_\alpha(n) K_{\alpha\beta}^{(W)}(n,m) \hat{\psi}_\beta(m), \tag{3.41}$$

where

$$\begin{aligned} K_{\alpha\beta}^{(W)} =& (\hat{M} + 4r)\delta_{nm}\delta_{\alpha\beta} \\ & - \frac{1}{2}\sum_\mu [(r - \gamma_\mu)_{\alpha\beta}\delta_{m,n+\hat{\mu}} + (r + \gamma_\mu)_{\alpha\beta}\delta_{m,n-\hat{\mu}}]. \end{aligned} \tag{3.42}$$

In this case, the two-point correlation function, in the continuum theory, takes the form

$$\langle \psi_\alpha(x)\bar{\psi}_\beta(y)\rangle = \lim_{a\to 0} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{[-i\gamma_\mu \tilde{\tilde{p}}_\mu + M(p)]_{\alpha\beta}}{\sum_\mu \tilde{\tilde{p}}_\mu^2 + M(p)^2} e^{ip\cdot(x-y)}, \tag{3.43}$$

where $\tilde{\tilde{p}}_\mu$ is defined as before and

$$M(p) = M + \frac{2r}{a}\sum_\mu \sin^2(\frac{p_\mu a}{2}). \tag{3.44}$$

This expression shows us that $M(p) \to M$ for any fixed value of p_μ as $a \to 0$. Also, near the edges of the Brillouin zone, $M(p)$ diverges as $a \to 0$. Thus, the fermion doubling problem is eliminated, though at the expense of breaking the chiral symmetry of the original action for $M = 0$ [11].

3.4 QED Action on the lattice

The action that describes QED in euclidean space, as we derived in the previous chapter is

$$S_{QED}^{(eucl.)} = S_G^{(eucl.)} + S_F^{(eucl.)} = \frac{1}{4}\int \mathrm{d}^4 x F_{\mu\nu}F_{\mu\nu} + \int \mathrm{d}^4 x \bar{\psi}(\gamma_\mu \mathcal{D}_\mu + M)\psi, \tag{3.45}$$

where M is the mass of the free Dirac field and $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative. We want to derive the lattice equivalent of this action, step by step. Firstly, we will start with the lattice action for free fermionic fields. To avoid the doubling problem, we will use Wilson fermions, and the corresponding action, written with dimensionless variables (for simplicity, we have dropped the hat over the fermionic variables), is

$$\begin{aligned} S_F^{(W)} &= (\hat{M} + 4r) \sum_n \bar{\psi}(n)\psi(n) \\ &\quad - \frac{1}{2} \sum_{n,\mu} [\bar{\psi}(n)(r - \gamma_\mu)\psi(n + \hat{\mu}) + \bar{\psi}(n + \hat{\mu})(r + \gamma_\mu)\psi(n)]. \end{aligned} \tag{3.46}$$

This action is invariant under the global transformations

$$\psi(n) \to G\psi(n), \tag{3.47}$$
$$\bar{\psi}(n) \to \bar{\psi}(n)G^{-1}, \tag{3.48}$$

where G is an element of the $U(1)$ group. Next, we demand that the theory is invariant under local $U(1)$ transformations, i.e., the group element G depends on the lattice site, $G(n)$. Since the Wilson action is non-diagonal (due to the derivatives that come up in the continuum limit), new degrees of freedom must be introduced. The group elements $G(n)$ do not act on Dirac indices, and therefore we will concentrate on a typical bilinear term, $\bar{\psi}(n)\psi(n + \hat{\mu})$.

In the continuum formulation, we know how a bilinear term like this one should be modified to be part of a gauge-invariant expression. The transformation of $\bar{\psi}(x)\psi(y)$, is as follows

$$\bar{\psi}(x)\psi(y) \to \bar{\psi}(x)G^{-1}(x)G(y)\psi(x),$$

but a gauge-potential-dependent factor must be included, that will compensate the gauge variation. This factor is known as the Schwinger line integral, and is given by

$$U(x, y) = e^{ie \int_x^y \mathrm{d}z_\mu A_\mu(z)}, \tag{3.49}$$

where the line integral is carried out along a path C, that connects x and y, and a summation over μ is implied. The line integral is also an element of the $U(1)$ group. Under a gauge transformation, $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \Lambda$, and $G(x) = e^{i\Lambda(x)}$, it is transformed as follows

$$U(x,y) \to G(x)U(x,y)G^{-1}(y), \tag{3.50}$$

We conclude that the bilinear expression in the fermion fields ψ and $\bar{\psi}$ that follows, is invariant under gauge transformations:

$$\bar{\psi}(x)U(x,y)\psi(y) = \bar{\psi}(x)e^{ie\int_x^y \mathrm{d}z_\mu A_\mu(z)}\psi(y). \tag{3.51}$$

If $y = x+\epsilon$, then the bilinears $\bar{\psi}(x)\psi(x+\epsilon)$ and $\bar{\psi}(x+\epsilon)\psi(x)$ are modified as follows:

$$\begin{aligned}
\bar{\psi}(x)\psi(x+\epsilon) &\to \bar{\psi}(x)U(x,x+\epsilon)\psi(x+\epsilon)\\
\bar{\psi}(x+\epsilon)\psi(x) &\to \bar{\psi}(x+\epsilon)U^\dagger(x,x+\epsilon)\psi(x)
\end{aligned}$$

where

$$U(x,x+\epsilon) = e^{ie\epsilon\cdot A(x)}$$

and $\epsilon \cdot A = \sum_\mu \epsilon_\mu A_\mu$.

Therefore, the substitutions needed to derive a gauge-invariant expression for the fermionic action on the lattice, are

$$\begin{aligned}
\bar{\psi}(n)(r-\gamma_\mu)\psi(n+\hat{\mu}) &\to \bar{\psi}(n)(r-\gamma_\mu)U_{n,n+\hat{\mu}}\psi(n+\hat{\mu})\\
\bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)\psi(n) &\to \bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)U_{n+\hat{\mu},n}\psi(n),
\end{aligned} \tag{3.52}$$

where

$$U_{n+\hat{\mu},n} = U^\dagger_{n,n+\hat{\mu}}, \tag{3.53}$$

and $U_{n,n+\hat{\mu}}$ is an element of the $U(1)$ gauge group and thus can take the form

$$U_{n,n+\hat{\mu}} = e^{i\phi_\mu(n)}, \tag{3.54}$$

where $\phi_\mu(n)$ is subject to the constraint $[0, 2\pi]$. The new action will be invariant under the following local transformations:

$$\begin{aligned}
&\psi(n) \to G(n)\psi(n),\\
&\bar{\psi}(n) \to \bar{\psi}(n)G^{-1}(n),\\
&U_{n,n+\hat{\mu}} \to G(n)U_{n,n+\hat{\mu}}G^{-1}(n+\hat{\mu}),\\
&U_{n+\hat{\mu},n} \to G(n+\hat{\mu})U_{n+\hat{\mu},n}G^{-1}(n).
\end{aligned} \tag{3.55}$$

Contrary to the matter fields mentioned before, the group elements $U_{n,n+\hat{\mu}}$ live on the links that connect two neighboring lattice sites, and thus, are referred to as link variables, or simply links. A schematic example is shown in fig. (3.1).

Figure 3.1: Schematic illustration of links on a lattice [11].

After making the above substitutions, the gauge-invariant action for Wilson fermions takes the form

$$\begin{aligned} S_F^{(W)}[\psi,\bar{\psi},U] = (\hat{M}+4r)\sum_n \bar{\psi}(n)\psi(n) \\ -\frac{1}{2}\sum_{n,\mu}[\bar{\psi}(n)(r-\gamma_\mu)U_{n,n+\hat{\mu}}\psi(n+\hat{\mu}) \\ +\bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)U^\dagger_{n,n+\hat{\mu}}\psi(n)]. \end{aligned} \tag{3.56}$$

We want to check that, in the continuum limit, the above action will take the form of equation (3.45), and so, we want to find the relation between the links and the vector potential $A_\mu(n)$. The procedure to do this is as follows: The vector potential $A_\mu(n)$ at the lattice site n has a real value and has a Lorentz index, and the same holds for $\phi_\mu(n)$, that parametrizes the link. However, the parameter $\phi_\mu(n)$ is restricted in the interval $[0, 2\pi]$, while the vector potential $A_\mu(n)$, in the continuum formulation, can take any real value. Since A_μ has the dimensions of inverse length, and ϕ_μ is dimensionless, we can make the anzatz $\phi_\mu(n) = caA_\mu(n)$, where a is the lattice spacing and c is a constant that will be determined. For $a \to 0$, the range of the vector potential will be infinite and the different ranges between A_μ and ϕ_μ are no longer a problem. Now, we rescale the fields and variables back to their canonical dimensions ($\hat{M} \to aM$, $\psi \to a^{3/2}\psi$, $\bar{\psi} \to a^{3/2}\bar{\psi}$) and we replace $U_{n,n+\hat{\mu}}$ for small values of the lattice spacing with the naive expansion

$$U_{n,n+\hat{\mu}} \approx 1 + icaA_\mu(n).$$

Then, the fermionic part of the action reduces, as it should, to the corresponding part

of the continuum action, if we choose $c = e$. We can also define a new notation as follows

$$U_\mu(n) \equiv U_{n,n+\hat{\mu}} = e^{ieaA_\mu(n)}, \tag{3.57}$$

and we can verify that $U_\mu(n)$ transforms under gauge-transformations in the following way

$$U_\mu(n) \to G(n)U_\mu(n)G^{-1}(n+\hat{\mu}) = e^{ieaA^G_\mu(n)},$$

where $A^G_\mu(n)$ is a discretized version of $A_\mu(x) - \frac{1}{e}\partial_\mu\Lambda(x)$. Now, to complete the lattice formulation of QED, we must also derive the lattice version of the kinetic term of the gauge fields. This term should be, again, strictly gauge-invariant and depend only on the link variables. Such functionals are constructed easily by taking the product of link variables around closed loops on the euclidean space-time lattice. Moreover, since the continuum version of this part of the Lagrangian is local, the loops should be as small as possible. Thus we consider the product of links around an elementary plaquette, that lies for example, in the $\mu - \nu$ plane. We define

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U^\dagger_\mu(n+\hat{\nu})U^\dagger_\nu(n), \tag{3.58}$$

where we have path-ordered the links. Note that this path ordering is irrelevant in the abelian case; however, the same does not hold in the non-abelian case. An example of an elementary plaquette is shown in fig. (3.2).

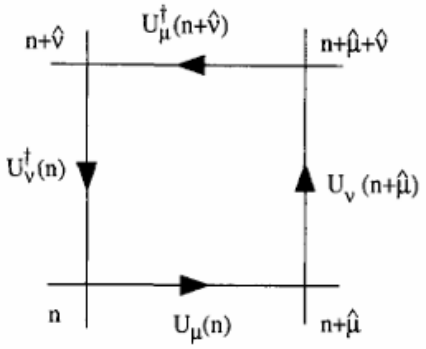

Figure 3.2: Illustration of an elementary plaquette $U_{\mu\nu}(n)$ with base at n lying in the $\mu\nu$ plane [11].

After inserting equation (3.57) into equation (3.58), one obtains

$$U_{\mu\nu}(n) = e^{iea^2 F_{\mu\nu}(n)}, \tag{3.59}$$

where $F_{\mu\nu}(n)$ is a discretized version of the continuum field tensor:

$$F_{\mu\nu}(n) = \frac{1}{a}[(A_\nu(n+\hat{\mu}) - A_\nu(n)) - (A_\mu(n+\hat{\nu}) - A_\mu(n))].$$

Immediately, one can write that for small lattice spacing

$$\frac{1}{e^2}\sum_n \sum_{\substack{\mu,\nu \\ \mu<\nu}} [1 - \frac{1}{2}(U_{\mu\nu}(n) + U^\dagger_{\mu\nu}(n))] \approx \frac{1}{4}\sum_{\substack{n \\ \mu,\nu}} a^4 F_{\mu\nu}(n) F_{\mu\nu}(n)$$

where the sum on the left-hand side extends over the contributions coming from all distinct plaquettes on the lattice but on the right-hand side, the sum extends over all μ and ν. The lattice action for the gauge potential can be written as follows

$$S_G[U] = \frac{1}{e^2}\sum_P [1 - \frac{1}{2}(U_P + U_P^\dagger)], \tag{3.60}$$

where U_P is called a plaquette variable and stands for the product of links around the boundary of a plaquette "P" i.e. in the counterclockwise direction.

For Wilson fermions, the lattice action for QED, takes the form

$$\begin{aligned} S_{QED}[U,\psi,\bar{\psi}] &= \frac{1}{e^2}\sum_P [1 - \frac{1}{2}(U_P + U_P^\dagger)] + (\hat{M} + 4r)\sum_n \bar{\psi}(n)\psi(n) \\ &- \frac{1}{2}\sum_{n,\mu} [\bar{\psi}(n)(r - \gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}) + \bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)U^\dagger_\mu(n)\psi(n)]. \end{aligned} \tag{3.61}$$

This action can be used in path-integral formulation, so that we can compute any correlation function of the fermionic and link variables. Since we want to ensure that the gauge invariance will not be destroyed when computing any path integrals, we must define a gauge-invariant integration measure (even though this is trivial in the abelian case). By parametrizing $U_\mu(n) = e^{i\phi_\mu(n)}$, the gauge-invariant measure to be used in a path integral expression is hence given by

$$DU = \prod_{n,\mu} \mathrm{d}\phi_\mu(n), \tag{3.62}$$

and the correlation functions of the Dirac fields and links can be calculated from the following path integral expression

$$\begin{aligned}\langle \psi_\alpha(n)\dots\bar{\psi}_\beta(m)\dots U_\mu(N)\dots\rangle = \\ = \frac{\int DUD\bar{\psi}D\psi(\psi_\alpha(n)\dots\bar{\psi}_\beta(m)\dots U_\mu(N)\dots)e^{-S_{QED}}}{\int DUD\bar{\psi}D\psi e^{-S_{QED}}}.\end{aligned} \tag{3.63}$$

These correlation functions depend on the parameters $\hat{M}$ and e, that are included in the fermionic and gauge lattice actions. In the interacting quantum theory, we can no longer say that these parameters correspond to the physical fermion mass and charge, and they must be handled as bare parameters, that have no direct physical meaning.

The lattice formulation of QED was presented in this section as it is described by Rothe (2005) [11].

3.5 QCD Action on the lattice

The action that describes continuum Quantum Chromodynamics in euclidean space, is

$$S_{QCD}^{(eucl.)} = S_G^{(eucl.)} + S_F^{(eucl.)} = \frac{1}{4}\int \mathrm{d}^4x F_{\mu\nu}^a F_{\mu\nu}^a + \int \mathrm{d}^4x \underset{\sim}{\bar{\psi}}(x)(\gamma_\mu \mathcal{D}_\mu + M_0)\underset{\sim}{\psi}(x). \tag{3.64}$$

where M_0 is the bare mass parameter, $\mathcal{D}_\mu = \partial_\mu + ig_0 \underset{\sim}{A}_\mu$ is the covariant derivative and g_0 is the bare coupling constant. In this section, we want to find the discretized version of this action. We start once again by the free fermionic action, in its discretized version, using Wilson fermions. In this case, we suppose that we have three free Dirac fields ψ^a, $(a = 1, 2, 3)$. The fermionic action takes the form (where we have dropped the labels reminding us of the euclidean formulation):

$$\begin{aligned}S_F = (\hat{M}_0 + 4r)\sum_n \sum_{a=1}^{3} \bar{\psi}^a(n)\psi^a(n) \\ - \frac{1}{2}\sum_{n,\mu}\sum_{a=1}^{3}[\bar{\psi}^a(n)(r-\gamma_\mu)\psi^a(n+\hat{\mu}) + \bar{\psi}^a(n+\hat{\mu})(r+\gamma_\mu)\psi^a(n)].\end{aligned} \tag{3.65}$$

The procedure to find the action will be similar to the abelian case, but in this

case, we replace the fields ψ and $\bar{\psi}$ with 3-component vectors and the link variables by the corresponding group elements of $SU(3)$ in the fundamental representation (3-dimensional). Now, we drop the tilde under the vectors and matrices, and to distinguish them from the components, we note that the components will be denoted by an index. The above action is invariant under the global transformations

$$\begin{aligned}\psi(x) &\to G\psi(x)\\ \bar{\psi}(x) &\to \bar{\psi}(x)G^{-1},\end{aligned}$$

where G is an element of $SU(3)$. We introduce the matrix-valued link variables $U_\mu(n)$, which we can write as

$$U_\mu(n) = e^{i\phi_\mu(n)}, \tag{3.66}$$

where $\phi_\mu(n)$ is a hermitian matrix that belongs to the Lie algebra of $SU(3)$. Then, the gauged fermionic action takes the form

$$\begin{aligned}S_F^{(W)} =&(\hat{M}_0 + 4r)\sum_n \bar{\psi}(n)\psi(n)\\ &- \frac{1}{2}\sum_{n,\mu}[\bar{\psi}(n)(r-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu})+\\ &+\bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)U_\mu^\dagger(n)\psi(n)].\end{aligned} \tag{3.67}$$

A typical term in the above part of the action can be written explicitly as

$$\bar{\psi}(n)(r-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}) = \sum_{\alpha,\beta,a,b}\bar{\psi}^a_\alpha(n)(r-\gamma_\mu)_{\alpha\beta}(U_\mu(n))_{ab}\psi^b_\beta(n+\hat{\mu}).$$

The gauged fermionic action is invariant under the local transformations:

$$\begin{aligned}&\psi(n) \to G(n)\psi(n),\\ &\bar{\psi}(n) \to \bar{\psi}(n)G^{-1}(n),\\ &U_\mu(n) \to G(n)U_\mu(n)G^{-1}(n+\hat{\mu}),\\ &U_\mu^\dagger(n) \to G(n+\hat{\mu})U_\mu^\dagger(n)G^{-1}(n)\end{aligned} \tag{3.68}$$

where $G(n)$ is an element of $SU(3)$ in the fundamental representation and thus, can be written in the form

$$G(n) = e^{i\Lambda(n)}, \tag{3.69}$$

where $\Lambda(n)$ is a hermitian matrix that belongs to the Lie algebra of $SU(3)$.

Now, we want to construct the rest of the action and we want it to be gauge-invariant as well. The simplest gauge-invariant quantity we can construct from the group elements $U_\mu(n)$, is the trace of the path ordered product of links along the boundary of an elementary plaquette. This is a generalization of the abelian case and can be written in the form

$$U_{\mu\nu}(n) = U_\mu(n)U_\nu(n+\hat{\mu})U_\mu^\dagger(n+\hat{\nu})U_\nu^\dagger(n). \tag{3.70}$$

Since the group elements do not commute, the order of the links in the product matters, in contrast to the abelian case. The new part of the action, that corresponds to the kinetic term and the self-interaction term for the gauge fields, takes the form

$$S_G = c\,\mathrm{Tr}\sum_{\substack{n\\ \mu<\nu}}[1 - \frac{1}{2}(U_{\mu\nu}(n) + U_{\mu\nu}^\dagger(n))], \tag{3.71}$$

where c is a constant that needs to be calculated.

Now, we want to check that this action reduces to equation (3.64) in the continuum limit. The dimensioned matrix-valued lattice field $A_\mu(n)$ is defined by

$$\phi_\mu(n) = g_0 a A_\mu(n). \tag{3.72}$$

In this case, $\phi_\mu(n)$ is defined in equation (3.66) and a is the lattice spacing. As we have mentioned before, $A_\mu(n)$ is an element of the Lie algebra of $SU(3)$ and is of the form

$$A_\mu(n) = \sum_{a=1}^{8} A_\mu^a(n)\frac{\lambda^a}{2}, \tag{3.73}$$

where $A_\mu^a(n)$ are eight real-valued vector fields corresponding to the eight generators of $SU(3)$, λ^a. After applying naive expansion to the links, and rescaling $\hat{M}_0$, ψ and $\bar{\psi}$ back to their canonical dimensions, the fermionic action takes the correct form in the continuum limit ($a \to 0$):

$$S_F^{(cont.)} = \int \mathrm{d}^4x\bar{\psi}(x)(\gamma_\mu(\partial + ig_0A_\mu) + M_0)\psi(x). \tag{3.74}$$

Next, we will check the continuum limit of the gauge part of the action. Therefore, we define the matrix-valued lattice field $\mathcal{F}_{\mu\nu}$ in the following way

$$U_{\mu\nu}(n) = e^{ig_0 a^2 \mathcal{F}_{\mu\nu}}. \tag{3.75}$$

Due to the fact that links do not commute the relation between $\mathcal{F}_{\mu\nu}$ and $A_\mu(n)$ is more complicated than in the abelian case. To find this relation, we use the Baker-Campbell-Hausdorff formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\ldots} \tag{3.76}$$

where the dots denote an infinite number of terms. Since $\phi_\mu(n)$ is proportional to the lattice spacing, we will keep only a few terms of the naive expansion of the exponential, as the others will vanish in the continuum limit. We use relations such as the following

$$\phi_\mu(n+\nu) \approx \phi_\mu(n) + a\partial_\nu\phi_\mu(n) + \cdots \approx g_0 a A_\mu(n) + g_0 a^2 \partial_\nu A_\mu(n) + \cdots ,$$

to find that

$$\mathcal{F}_{\mu\nu} \xrightarrow[a\to 0]{} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_0[A_\mu, A_\nu]. \tag{3.77}$$

which is the result we expect in the continuum limit. Since, as we mentioned before, $F_{\mu\nu}$ is an element of the Lie algebra of $SU(3)$, we can write it as

$$F_{\mu\nu} = \sum_{a=1}^{8} F^a_{\mu\nu} \frac{\lambda^a}{2}, \tag{3.78}$$

and the eight components of $F^a_{\mu\nu}$ and A^a_μ follow the relation

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_0 f_{abc} A^b_\mu A^c_\nu. \tag{3.79}$$

Then, for small lattice spacing, we approximate the first two non-trivial terms in the expansion of the exponential and we find that the gauge action takes the following form in the continuum limit

$$S_G \to c\frac{g_0^2}{2}\left(\frac{1}{2}\,\mathrm{Tr}\int \mathrm{d}^4x F_{\mu\nu}F_{\mu\nu}\right), \tag{3.80}$$

where a sum over μ and ν is understood and the term in parentheses is the gauge part of the continuum action we derived in the previous chapter. Thus, we must choose $c = 2/g_0^2$. If we had N (rather than 3) Dirac fields, for all $N > 1$, the gauge part of the action would be given by

$$S_G^{(SU(N))} = \beta \sum_P [1 - \frac{\mathrm{Tr}}{2N}(U_P + U_P^\dagger)], \tag{3.81}$$

where

$$\beta = \frac{2N}{g_0^2}. \tag{3.82}$$

Just like in the abelian case, the sum extends over all distinct plaquettes on the lattice, and we note that U_P is the path-ordered product of links around the boundary of a plaquette P. To ensure that the action is hermitian, both orientations for this product are considered.

To define the quantum theory, we also need to specify the path integral expression that will be used to calculate any correlation functions. This expression will be similar to the abelian case, but now the fields ψ_α, $\bar{\psi}_\alpha$ and $U_\mu(n)$ will carry extra "color" indices that refer to the three-colored quarks lying in the fundamental representation of $SU(N)$. In addition, the integration measure DU will depend on the eight real variables that parametrize the group elements of $SU(3)$, and the integration will be performed over the group manifold. Since we do not want the quantum fluctuations to destroy the gauge invariance, the integration measure must also obey it. We write $\alpha_l^a (a = 1, \cdots, 8)$ the group parameters on which the l'th link depends, the integration measure, also known as the Haar measure, must be written as

$$DU = \prod_l J(\alpha_l)(\mathrm{d}\alpha_l), \tag{3.83}$$

where α_l (l runs over all links of the lattice) stands for the set $\{\alpha_l^a\}$, and

$$(\mathrm{d}\alpha_l) \equiv \prod_{a=1}^{8} \alpha_l^a \tag{3.84}$$

The Jacobian $J(\alpha_l)$ can be determined by requesting gauge invariance.

Any correlation function that involves fermionic fields and link variables is to be computed from the following path integral expression

$$\begin{aligned}&\langle\psi^{a_1}_{\alpha_1}(n_1)\cdots\bar{\psi}^{b_1}_{\beta_1}(m_1)\cdots U^{cd}_{\mu_1}(k_1)\cdots\rangle = \\ &= \frac{1}{Z}\int DUD\bar{\psi}D\psi\psi^{a_1}_{\alpha_1}(n_1)\cdots\bar{\psi}^{b_1}_{\beta_1}(m_1)\cdots U^{cd}_{\mu_1}(k_1)\cdots e^{-S_{QCD}},\end{aligned} \tag{3.85}$$

where

$$Z = \int DUD\bar{\psi}D\psi e^{-S_{QCD}}, \tag{3.86}$$

and the lattice action for QCD, for the case of Wilson fermions is given by

$$\begin{aligned}S_{QCD} &= S_G[U] + S_F^{(W)}[U,\psi,\bar{\psi}] \\ &= \frac{6}{g_0^2}\sum_P[1-\frac{\mathrm{Tr}}{6}(U_P+U_P^\dagger)] + (\hat{M}_0+4r)\sum_n\bar{\psi}(n)\psi(n) \\ &- \frac{1}{2}\sum_{n,\mu}[\bar{\psi}(n)(r-\gamma_\mu)U_\mu(n)\psi(n+\hat{\mu}) + \bar{\psi}(n+\hat{\mu})(r+\gamma_\mu)U_\mu^\dagger(n)\psi(n)].\end{aligned} \tag{3.87}$$

The lattice formulation of QCD was presented in this section as it is described by Rothe (2005) [11].

3.6 Perturbation Theory and the example of the ϕ^3 theory

Since QCD is asymptotically free, we can use weak coupling perturbation theory in continuum QCD to look into its short distance properties. However, the lattice formulation of QCD was introduced to study non-perturbative phenomena, like quark confinement, hadron masses, etc. Nevertheless, perturbative calculations are very important for determining renormalization coefficients, relating lattice Green functions to physical ones, especially in cases where non-perturbative estimates of these coefficients are not available or sufficiently accurate. In the lattice formulation, "irrelevant" vertices will be included that do not translate to the continuum. These vertices vanish in the continuum limit, but a finite number of them will contribute in a given order of perturbation theory to the Green functions, in the limit $a \to 0$. Therefore, we need to consider a larger number of Feynman diagrams. Also, due to

the fact that the integrands are now periodic functions of the momenta, it is not clear that we can renormalize lattice gauge theories [11].

Since we have introduced a space-time lattice as a regulator for the quantum theory, we have manifestly broken Poincaré invariance. We want to find out if the symmetry is restored in the continuum limit, but a non-perturbative analysis of this is very difficult. By working in a perturbative way, we can actually answer this question. Moreover, there are symmetries that we know are broken on a quantum level. This is an anomaly, which is reproduced correctly with the lattice formulation [11].

These are just some of the reasons we are interested in a perturbative treatment of lattice gauge theories. In the rest of this section, we will discuss a scalar field theory to show how the integrands of Feynman integrals are changed by lattice artifacts, as presented by Rothe (2005) [11]. The ϕ^3-theory is the simplest non-trivial example and it can be expanded to apply to any lattice field theory relevant to elementary particle physics.

Weak Coupling Expansion of Correlation Functions in the ϕ^3-Theory

We consider the euclidean continuum action for a real scalar field

$$S[\phi] = \frac{1}{2}\int \mathrm{d}^4x\phi(x)(-\Box + M^2)\phi(x) + \frac{g_0}{3!}\int \mathrm{d}^4x(\phi(x))^3, \tag{3.88}$$

In this action, $\Box$ is the four-dimensional Laplacian, M is the bare mass parameter and g_0 is the bare coupling parameter, which has dimensions of mass. We introduced the combinatorial factor 3! to simplify the Feynman rules. We transition to a space-time lattice and we scale ϕ, $\Box$, M and g_0 with the lattice spacing a according to their canonical dimensions, that is, we define the corresponding dimensionless quantities: $\hat{\phi} = a\phi$, $\hat{M} = aM$, $\hat{\Box} = a^2\Box$ and $\hat{g}_0 = ag_0$. Then, the lattice action takes the form

$$S[\hat{\phi}] = \sum_{n,m}\hat{\phi}_n K_{nm}\hat{\phi}_m + \frac{\hat{g}_0}{3!}\sum_n \hat{\phi}_n, \tag{3.89}$$

where n and m are 4-component vectors labeling the lattice sites and K is the same matrix that appears in the Klein-Gordon case on the lattice and is defined as

$$K_{nm} = -\sum_{\mu>0}[\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{nm}] + \hat{M}^2\delta_{nm}$$

Correlation functions of the fields $\hat{\phi}_n$ are expected to be computed from the functional

$$Z[\hat{J}] = \int D\hat{\phi} e^{-S[\hat{\phi}]+\sum \hat{J}_n \hat{\phi}_n}$$

by differentiating this expression with respect to $\hat{J}_n$:

$$\langle \hat{\phi}_{n_1} \hat{\phi}_{n_2} \cdots \hat{\phi}_{n_l} \rangle = \frac{1}{Z[0]} \left\{ \frac{\partial^l Z[\hat{J}]}{\partial \hat{J}_{n_1} \cdots \partial \hat{J}_{n_l}} \right\}_{J=0} \tag{3.90}$$

In perturbation theory, $Z[\hat{J}]$ can be computed as follows:

$$\begin{aligned} Z[\hat{J}] &= e^{-S_{int}[\frac{\partial}{\partial \hat{J}}]} Z_0[\hat{J}] \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \left(-S_{int}[\frac{\partial}{\partial \hat{J}}] \right)^N Z_0[\hat{J}] \end{aligned} \tag{3.91}$$

where

$$S_{int}[\hat{\phi}] = \frac{\hat{g}_0}{3!} \sum_n \hat{\phi}_n^3 \tag{3.92}$$

and $S_{int}[\partial/\partial\hat{J}]$ is obtained by substituting $\hat{\phi}_n \to \partial/\partial\hat{J}_n$. $Z_0[\hat{J}]$ is the generating functional of the free theory

$$Z_0[\hat{J}] = \frac{1}{\sqrt{\det K}} e^{\frac{1}{2}\sum_{n,m} \hat{J}_n K^{-1}_{nm} \hat{J}_m}. \tag{3.93}$$

Now, we can derive the Feynman rules. In every given order of perturbation theory, the contribution to the correlation function can be represented by a set of Feynman diagrams constructed from the interaction vertices with coupling $-\hat{g}_0$ and propagators $\hat{\Delta}_{nm} = K^{-1}_{nm}$. We have two types of propagators (lines) that can be found in a Feynman diagram:

1. lines that connect external lattice sites that appear in the correlation function with an interaction vertex, called external lines, and

2. lines that connect two vertices, called internal lines.

To understand this better, we will examine the contribution of order g_0^2 to the two-point correlation function $\langle \phi_n \phi_m \rangle$, which is given by

$$\langle\hat{\phi}_n\hat{\phi}_m\rangle = \frac{1}{2}\hat{g}_0^2\sum_{l,l'}\hat{\Delta}_{nl}\hat{\Pi}_{ll'}\hat{\Delta}_{l'm}, \tag{3.94}$$

where

$$\hat{\Pi}_{ll'} = (\hat{\Delta}_{ll'})^2. \tag{3.95}$$

and the term 1/2 is a "symmetry factor" that can be calculated for every Feynman diagram. Now, we will transition to momentum space, where the free propagator $\hat{\Delta}_{nm}$ has the following Fourier representation

$$\hat{\Delta}_{nm} = \int_{-\pi}^{\pi}\frac{\mathrm{d}^4\hat{k}}{(2\pi)^4}\frac{e^{i\hat{k}\cdot(n-m)}}{\widehat{k^2}+\hat{M}^2}, \tag{3.96}$$

where

$$\widehat{k^2} = \sum_{\mu=1}^{4}\widehat{k_\mu^2}, \tag{3.97}$$

and

$$\widehat{k_\mu} = 2\sin\frac{\hat{k}_\mu}{2}. \tag{3.98}$$

Because the integral of the free propagator contains the square of $\widehat{k_\mu}$ (which is a dimensionless periodic function), the integrand is a periodic function of $\hat{k}_\mu$ (momentum measured in lattice units) with periodicity 2π. After we insert the expression for the free propagator in the two-point function, and perform the sum over l and l' using the Fourier representation for the periodic δ-function, we arrive at

$$\langle\hat{\phi}_n\hat{\phi}_m\rangle = \int_{BZ}\frac{\mathrm{d}^4\hat{k}}{(2\pi)^4}\frac{\mathrm{d}^4\hat{k}}{(2\pi)^4}\hat{G}(\hat{k},\hat{k}';\hat{M})e^{i\hat{k}\cdot n - i\hat{k}'\cdot m}, \tag{3.99}$$

where

$$\hat{G}(\hat{k},\hat{k}';\hat{M}) = \frac{1}{\widehat{k^2}+\hat{M}^2}\hat{\Pi}(\hat{k},\hat{k}';\hat{M})\frac{1}{\widehat{k'^2}+\hat{M}^2}, \tag{3.100}$$

and

$$\hat{\Pi}(\hat{k},\hat{k}';\hat{M}) = \frac{\hat{g}_0^2}{2}\int_{BZ}\prod_{i=1}^{2}\frac{\mathrm{d}^4\hat{l}_i}{(2\pi)^4}[(2\pi)^4]^2\delta_P^{(4)}(\hat{k}-\hat{l}_1-\hat{l}_2)\delta_P^{(4)}(\hat{k}'-\hat{l}_1-\hat{l}_2)\prod_{i=1}^{2}\frac{1}{(\widehat{l_i^2}+\hat{M}^2)}. \tag{3.101}$$

In this expression, $\hat{l}_i$ are the "line" momenta carried by the internal lines of the momentum Feynman diagram. The general structure of these expressions looks like

that of the continuum formulations, if $\hat{k}_\mu$ replaces $\widehat{k_\mu}$, except that all the variables are in lattice units. Due to the periodic delta functions, momenta are consered at the vertices modulo $2n\pi$ (where n is an integer). Since we want the momenta to belong in the Brillouin zone, the integration over $\hat{l}_2$ is equivalent to setting $\hat{l}_2 = \hat{k} - \hat{l}_1$, which is the same momentum conservation that comes up in the continuum formulation. Now we are left with

$$\hat{\Pi}(\hat{k}, \hat{k}'; \hat{M}) = (2\pi)^4 \delta_P^{(4)}(\hat{k} - \hat{k}') \hat{\Pi}(\hat{k}, \hat{M}),$$

where

$$\hat{\Pi}(\hat{k}, \hat{M}) = \frac{\hat{g}_0^2}{2} \int_{BZ} \frac{\mathrm{d}^4 \hat{q}}{(2\pi)^4} \frac{1}{[\widehat{q^2} + \hat{M}^2][\widehat{(k-q)^2} + \hat{M}^2]},$$

and

$$\widehat{(k-q)}_\mu = 2 \sin[\frac{1}{2}(\hat{k} - \hat{q})_\mu].$$

Next, we want to compute this contribution of the diagram to the physical two-point function $\langle \phi(x)\phi(y) \rangle$, so we rescale the dimensionless variables back to their physical dimensions. The dimensioned variables are defined as $\phi = \hat{\phi}/a,\ x = na,\ y = ma,\ M = \hat{M}/a,\ k = \hat{k}/a,\ k' = \hat{k}'/a,\ q = \hat{q}/a$ and $g_0 = \hat{g}_0/a$ and we will see how the integral behaves as we decrease the lattice spacing and we keep $x,\ y,\ M$ and g_0 fixed. This contribution can be formally written as

$$\langle \phi(x)\phi(y) \rangle = \lim_{a \to 0} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{d}^4 k'}{(2\pi)^4} G(k, k'; M, a) e^{ik \cdot x} e^{-ik' \cdot y}, \tag{3.102}$$

where

$$G(k, k'; M, a) = (2\pi)^4 \delta_P^{(4)}(k - k') \left[\frac{1}{\tilde{k^2} + M^2} \Pi(k; M, a) \frac{1}{\tilde{k^2} + M^2} \right], \tag{3.103}$$

and

$$\Pi(k; M, a) = \frac{g_0^2}{2} \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{[\tilde{q}^2 + M^2][\widetilde{(k-q)}^2 + M^2]}. \tag{3.104}$$

The generic definition of the "tilded" dimensioned quantities is

$$\begin{aligned} \tilde{p}_\mu &= \frac{2}{a} \sin \frac{p_\mu a}{2} \\ \tilde{p}^2 &= \sum_{\mu=1}^{4} \tilde{p}_\mu^2 \end{aligned} \tag{3.105}$$

Graphically, the quantity appearing in square brackets in Eq. (3.103) is given in fig. (3.3).

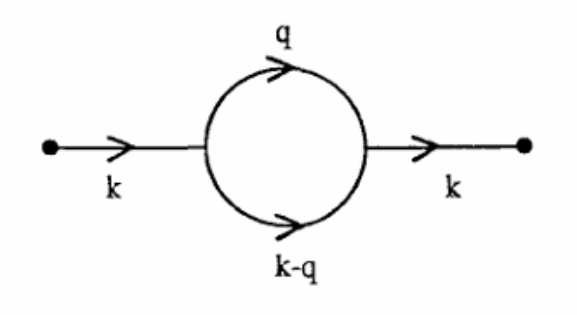

Figure 3.3: Contribution of $\mathcal{O}(g_0^2)$ to the propagator in momentum space [11].

In the continuum formulation, the Feynman rules that govern each Feynman diagram are the following [10]:

1. For each propagator, we have a line that mathematically will be given by $\frac{1}{q^2+M^2}$;
2. For each vertex, we will have a new coupling contribution, $-g_0$;
3. For each external line, we have an exponential factor $e^{ik\cdot x}$
4. In each vertex, we impose momentum conservation
5. We perform integration over each internal momentum: $\int \frac{\mathrm{d}^4 q}{(2\pi)^4}$;
6. Lastly, we divide by the summetry factor.

In relation to the continuum formulation, these are the important properties of the Fourier transform of the above correlation function computed on the lattice [11]:

1. The general structure of $G(k, k'; M, a)$ does not change in respect to the continuum formulation, except that propagators are replaced by their lattice analogues,
$$\Delta(p) = \frac{1}{\tilde{p}^2 + M^2}, \tag{3.106}$$
and we integrate the momenta over the Brillouin zone $[-\frac{\pi}{a}, \frac{\pi}{a}]^4$. These are the only changes applied to the Feynman rules of the continuum formulation.
2. The lattice propagators reduce to their continuum analogue when taking the limit $a \to 0$.

3. $G(k, k'; M, a)$ is a periodic function in each of the components of the momenta with periodicity $2\pi/a$.

4. The integrand of the lattice Feynman integral (eq. (3.104)) is a periodic function of the loop momentum q, with periodicity $2\pi/a$, and it possesses a finite continuum limit.

5. If this integrand (eq. (3.104)) is replaced by its naive continuum limit, the new integral is given by the continuum Feynman rules with a momentum cutoff π/a.

These properties hold for any Feynman diagram if an appropriate set of loop integration variables is chosen. A subset of the line momenta is a natural set of integration variables.

If we have a Feynman integral that has the above properties, we want to explore its continuum limit. Generally, we cannot calculate this limit by evaluating the integrals for finite lattice spacing and then taking the limit $a \to 0$. However, since the integrands are finite in this limit, we can expect, under certain conditions, to be able to replace them with their naive continuum limit which does not have the periodic structure that complicates the calculations [11].

Chapter 4

Supersymmetry

Supersymmetric Action in the continuum

In Supersymmetry (SUSY), each fermionic field is associated with a bosonic field and vice versa, which is known as its supersymmetric partner. The spins of the two supersymmetric partners differ by a half-integer, but they have all the other quantum numbers and mass equal.

We have studied a Supersymmetric version of QCD (SQCD). This theory serves as a prototype model for complex supersymmetric theories; while being simpler than supersymmetric extensions of the Standard Model, it shares many features with it (with a notable exception of SQCD not containing chiral fields), and it thus presents itself as an ideal forerunner for Standard Model and Beyond the Standard Model (BSM) investigations [4]. Other superymmetric models which have been studied on the lattice thus far cannot easily be generalized: e.g. the Wess-Zumino model contains no gauge fields and, supersymmetric Yang-Mills contains no matter fields.

Even though it is not the only choice for the supersymmetric QCD (SQCD) action, we chose to express it in the Wess-Zumino (WZ) gauge. In this case, auxiliary fields are eliminated from the action, and only the physical fields SQCD remain: the gluon together with the gluino (which has spin 1/2); and for each flavor a Dirac fermion (quark) and two squarks (which are complex scalars).

In the continuum and in Minkowski space, the action of SQCD is:

$$\begin{aligned}\mathcal{S}_{SQCD} = \int d^4x \Big[& -\frac{1}{4}u^{\alpha}_{\mu\nu}u^{\alpha\mu\nu} + \frac{i}{2}\bar{\lambda}^{\alpha}\gamma^{\mu}\mathcal{D}_{\mu}\lambda^{\alpha} \\ & - \mathcal{D}_{\mu}A^{\dagger}_{+}\mathcal{D}^{\mu}A_{+} - \mathcal{D}_{\mu}A_{-}\mathcal{D}^{\mu}A^{\dagger}_{-} + i\bar{\psi}\gamma^{\mu}\mathcal{D}_{\mu}\psi \\ & - i\sqrt{2}g(A^{\dagger}_{+}\bar{\lambda}^{\alpha}T^{\alpha}P_{+}\psi - \bar{\psi}P_{-}\lambda^{\alpha}T^{\alpha}A_{+} + A_{-}\bar{\lambda}^{\alpha}T^{\alpha}P_{-}\psi - \bar{\psi}P_{+}\lambda^{\alpha}T^{\alpha}A^{\dagger}_{-}) \\ & - \frac{1}{2}g^2(A^{\dagger}_{+}T^{\alpha}A_{+} - A_{-}T^{\alpha}A^{\dagger}_{-})^2 + m(\bar{\psi}\psi - mA^{\dagger}_{+}A_{+} - mA_{-}A^{\dagger}_{-})\Big],\end{aligned} \tag{4.1}$$

where ψ is the quark field, u_μ is the gluon field, λ is the gluino field and $A_\pm$ are the squark field components; T^α are the generators of the SU($N_c = 3$) gauge group and $P_\pm$ are projectors: $P_\pm = (1 \pm \gamma_5)/2$. The quark and squark fields and their masses m, have an implicit color index and an implicit flavor index and a summation over repeated indices is intended [4]. We are also defining the covariant derivatives and the gluon tensor:

$$\begin{aligned}\mathcal{D}_{\mu}A_{+} &= \partial_{\mu}A_{+} + igu^{\alpha}_{\mu}T^{\alpha}A_{+} \\ \mathcal{D}_{\mu}A^{\dagger}_{-} &= \partial_{\mu}A^{\dagger}_{-} + igu^{\alpha}_{\mu}T^{\alpha}A^{\dagger}_{-} \\ \mathcal{D}_{\mu}A_{-} &= \partial_{\mu}A_{-} - igA_{-}T^{\alpha}u^{\alpha}_{\mu} \\ \mathcal{D}_{\mu}A^{\dagger}_{+} &= \partial_{\mu}A^{\dagger}_{+} - igA^{\dagger}_{+}T^{\alpha}u^{\alpha}_{\mu} \\ \mathcal{D}_{\mu}\psi &= \partial_{\mu}\psi + igu^{\alpha}_{\mu}T^{\alpha}\psi \\ \mathcal{D}_{\mu}\lambda &= \partial_{\mu}\lambda + ig[u_{\mu}, \lambda] \\ u_{\mu\nu} &= \partial_{\mu}u_{\nu} - \partial_{\nu}u_{\mu} + ig[u_{\mu}, u_{\nu}].\end{aligned} \tag{4.2}$$

This action is invariant under the following supersymmetric transformations:

$$\begin{aligned}\delta_{\xi}A_{+} &= -\sqrt{2}\bar{\xi}P_{+}\psi, \\ \delta_{\xi}A_{-} &= -\sqrt{2}\bar{\psi}P_{+}\xi, \\ \delta_{\xi}(P_{+}\psi) &= i\sqrt{2}(\mathcal{D}_{\mu}A_{+})P_{+}\gamma^{\mu}\xi - \sqrt{2}mP_{+}\xi A^{\dagger}_{-}, \\ \delta_{\xi}(P_{-}\psi) &= i\sqrt{2}(\mathcal{D}_{\mu}A_{-})^{\dagger}P_{-}\gamma^{\mu}\xi - \sqrt{2}mA_{+}P_{-}\xi, \\ \delta_{\xi}u^{\alpha}_{\mu} &= -i\bar{\xi}\gamma^{\mu}\lambda^{\alpha}, \\ \delta_{\xi}\lambda^{\alpha} &= \frac{1}{4}u^{\alpha}_{\mu\nu}[\gamma^{\mu}, \gamma^{\nu}]\xi - ig\gamma^{5}\xi(A^{\dagger}_{+}T^{\alpha}A_{+} - A_{-}T^{\alpha}A^{\dagger}_{-}).\end{aligned} \tag{4.3}$$

where ξ is a Grassmann parameter [4]. The fact that supersymmetric transformations are parametrized by a single Grassmann parameter ξ is because we are studying an $\mathcal{N} = 1$ version of SQCD.

Due to additional infinities that appear upon functionally integrating over orbits, we introduce a gauge-fixing term and a Faddeev-Popov term to the Lagrangian. As is the case in ordinary QCD, this Lagrangian is invariant under Becchi-Rouet-Stora-Tyutin (BRST) transformations, but it is no longer manifestly gauge invariant. With this procedure, it is guaranteed that the Green functions of any gauge invariant object will not depend on the gauge we choose, in every order of the perturbation theory. The Faddeev-Popov procedure, when choosing the following gauge-fixing term, in Euclidean space:

$$\mathcal{S}^E_{GF} = \frac{1}{\alpha}\int d^4x\,\mathrm{Tr}(\partial_\mu u_\mu)^2, \tag{4.4}$$

where α is the gauge parameter ($\alpha = 1$ in the Feynman gauge and $\alpha = 0$ in the Landau gauge), gives

$$\mathcal{S}^E_{Ghost} = -2\int d^4x\,\mathrm{Tr}(\bar{c}\partial_\mu \mathcal{D}_\mu c), \tag{4.5}$$

where the ghost field c is a Grassmann scalar which transforms in the adjoint representation of the gauge group and $\mathcal{D}_\mu c = \partial_\mu c + ig[u_\mu, c]$. Supersymmetry is broken by this gauge fixing term. In spite of that, since the renormalized theory is independent of the chosen gauge fixing term, and a lattice discretization of the SQCD action will necessarily break supersymmetry, we can adopt this standard covariant gauge fixing term [4].

Symmetries of the action

Parity $\mathcal{P}$ and Charge Conjugation $\mathcal{C}$ are symmetries of the continuum action, and in euclidean space they are defined in the following way:

$$\mathcal{P}: \begin{cases} u_0(x) \to u_0(x_\mathcal{P}),\ u_k(x) \to -u_k(x_\mathcal{P} - a\hat{k}),\ k = 1,2,3 \\ \psi(x) \to \gamma_0\psi(x_\mathcal{P}) \\ \bar{\psi}(x) \to \bar{\psi}(x_\mathcal{P})\gamma_0 \\ \lambda^\alpha(x) \to \gamma_0\lambda^\alpha(x_\mathcal{P}) \\ \bar{\lambda}^\alpha(x) \to \bar{\lambda}^\alpha(x_\mathcal{P})\gamma_0 \\ A_\pm(x) \to A^\dagger_\mp(x_\mathcal{P}) \\ A^\dagger_\pm(x) \to A_\mp(x_\mathcal{P}) \end{cases} \tag{4.6}$$

where $x_\mathcal{P} = (-\mathbf{x}, x_0)$.

$$\mathcal{C}: \begin{cases} u_\mu^\alpha T^\alpha \to -u_\mu^\alpha (T^\alpha)^*, \ \mu = 0,1,2,3 \\ \psi(x) \to -\mathcal{C}\bar{\psi}(x)^T \\ \bar{\psi}(x) \to \psi(x)^T \mathcal{C}^\dagger \\ \lambda(x) \to \mathcal{C}\bar{\lambda}(x)^T \\ \bar{\lambda}(x) \to -\lambda(x)^T \mathcal{C}^\dagger \\ A_\pm(x) \to A_\mp(x) \\ A_\pm^\dagger(x) \to A_\mp^\dagger(x) \end{cases} \tag{4.7}$$

where ^{T} means transpose. The matrix $\mathcal{C}$ satisfies: $(\mathcal{C}\gamma_\mu)^T = \mathcal{C}\gamma_\mu, \quad \mathcal{C}^T = -\mathcal{C}$ and $\mathcal{C}^\dagger\mathcal{C} = 1$. In four dimensions, in a standard basis for γ matrices, in which $\gamma_0, \ \gamma_2$ are symmetric and $\gamma_1, \ \gamma_3$ are anti-symmetric, $\mathcal{C} = -i\gamma_0\gamma_2$ [4].
The gluino is a Majorana fermion, and therefore, obeys the Majorana condition:

$$(\bar{\lambda}^\alpha)^T = \mathcal{C}\lambda^\alpha. \tag{4.8}$$

Majorana fermions have half the degrees of freedom compared to Dirac fermions. In the case of the gluino, their Majorana nature is compatible with the SUSY requirement, that the number of degrees of freedom for the fermionic and bosonic supersymmetric partners is the same.

Further symmetries in the massless case are [4]:

$U(1)_R$ rotates the quark and gluino fields in opposite direction:

$$\mathcal{R}: \begin{cases} \psi(x) \to e^{i\theta\gamma_5}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}(x)e^{i\theta\gamma_5} \\ \lambda(x) \to e^{-i\theta\gamma_5}\lambda(x) \\ \bar{\lambda}(x) \to \bar{\lambda}(x)e^{-i\theta\gamma_5} \end{cases} \tag{4.9}$$

The $\mathcal{R}$-symmetry does not commute with the SUSY generators.

$U(1)_A$ rotates the squark and quark fields in the same direction as follows:

$$\chi: \begin{cases} \psi(x) \to e^{i\theta'\gamma_5}\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}(x)e^{i\theta'\gamma_5} \\ A_\pm(x) \to e^{i\theta'}A_\pm(x) \\ A_\pm^\dagger(x) \to e^{-i\theta'}A_\pm^\dagger(x) \end{cases}$$

Chapter 5

Overlap Action

5.1 Nielsen-Ninomiya Theorem

Nielsen and Ninomiya (1981) studied in depth the assumption that when a fermion action does not lead to doublers, chirality is explicitly broken and vice versa. They then proved the following famous "no-go" theorem about doubling and chirality [9]. The assumptions of the theorem are:

- A quadratic fermion action $\bar{\psi}(x)iH(x-y)\psi(y)$, where H is Hermitian, has a Fourier transform $H(p)$ defined for all p in the Brillouin Zone (BZ), and has a continuous first derivative everywhere in the BZ. $H(p)$ should behave as $\gamma_\mu p_\mu$ in the limit of small p_μ.

- A local conserved charge $Q \equiv \sum_x j_0(x)$, where $j_0(x)$ is a function of the fermionic fields $\psi(y)$ where y is close to x.

- Q is quantized.

The theorem states that, if all these assumptions are true, $H(p)$ has the same number of left handed and right handed fermions for each eigenvalue of Q.

The theorem has often been interpreted incorrectly. This wrong interpretation says that we cannot find a lattice action that has no doublers, is chiral and "ultra-local" (that means that the couplings extend over a finite number of lattice spacings). Actually, the theorem is evaded by the quantization of the charge, and this is the constraint that some modern four-dimensional chiral actions relax [5].

Historically, lattice actions have been designed in a way that they are constrained by ultra-locality, but what we really need in order to theoretically describe a field properly, is locality. That means that we restrict the range of the action to be on the order

of the size of the spatial cutoff. If the lattice couplings in an action fall off exponentially with distance (in lattice spacing units; for example $S = \sum_{x,r} \bar{\psi}(x)C(r)\psi(x+r)$ with $C(r) \simeq \exp(-r/\xi)$, $\xi \propto a$), then this action corresponds to a local one in the continuum limit, while an action with couplings that fall off slower (i.e. power law) does not [5].

We can find a deeper connection between chirality and doubling when thinking about the anomaly. A theorem due to Adler states that we cannot regularize a single-fermion theory, so that it is consistent with chiral symmetry, and therefore, the axial current is conserved. With a lattice regulator, we have to adopt one of the following options: (i) add fermions to cancel the anomaly - *naive and staggered fermions*, (ii) explicitly break chiral symmetry - *Wilson fermions*, (iii) redefine what we mean by a chiral rotation - *overlap fermions, which are explained further in this chapter*, or (iv) change the dimensionality of space-time - *domain wall fermions* [5].

5.2 Overlap Operator

The overlap-Dirac operator that has been derived from the overlap formulation of chiral fermions on the lattice provides a lattice regularization of massless QCD in which fine tuning is not needed [1]. The overlap-Dirac operator D_o of a massless fermion preserves chiral symmetry on the lattice without fermion doubling, according to the Nielsen-Ninomiya theorem, and can be written as

$$D_o = \frac{1}{a}[1 + \gamma_5 O], \tag{5.1}$$

where a is the lattice spacing, O depends on the link variables and is Hermitian with eigenvalues ± 1. The simplest example for an overlap-Dirac operator is the Neuberger-Dirac operator

$$D_N = \frac{1}{a}c[1 + X(X^\dagger X)^{-1/2}], \tag{5.2}$$

$$X = D_W - \frac{1}{a}\rho, \tag{5.3}$$

where D_W is the Wilson-Dirac operator (the Wilson parameter is set $r = 1$)

$$D_W = \frac{1}{2}[\gamma_\mu(\nabla^*_\mu + \nabla_\mu) - a\nabla^*_\mu\nabla_\mu], \tag{5.4}$$

$$\nabla_\mu\psi(x) = \frac{1}{a}[U_\mu(x)\psi(x + a\hat{\mu}) - \psi(x)], \tag{5.5}$$

and ρ is a real parameter restricted by the condition $0 < \rho < 2$. We can absorb the normalization constant $c = 1$ in the definition of the fermion fields.

The Neuberger-Dirac operator D_N is not analytic in the link variables when the operator X has a vanishing eigenvalue, unlike the Wilson-Dirac operator D_W. Unfortunately, this happens because we put strictly massless fermions on the lattice. We expect no harmful effects due to the lack of analyticity in the continuum limit [1].

D_N satisfies the Ginsparg-Wilson relation [8]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D, \tag{5.6}$$

which protects the quark masses from additive renormalization. Because of the Ginsparg-Wilson relation, we can use, at finite lattice spacing, properties associated with chiral symmetry. Indeed, that means that a chiral symmetry exists as an exact symmetry of the lattice action under the modified chiral transformations

$$\delta\psi(x) = \gamma_5(1 - \frac{1}{2}aD)\psi(x), \quad \delta\bar{\psi}(x) = \bar{\psi}(x)(1 - \frac{1}{2}aD)\gamma_5. \tag{5.7}$$

Under flavor-singlet chiral transformations, the integral measure is not invariant and therefore an axial anomaly arises. Moreover, it has been proved that we can renormalize lattice gauge theories with Ginsparg-Wilson fermions to all orders of perturbation theory.

Since the Nielsen-Ninomiya theorem does not affect lattice Dirac operators that satisfy the Ginsparg-Wilson relation, these operators are safe from fermion doubling. Therefore, a lattice formulation of QCD that follows this relation would solve the issues of the standard method (i.e. Wilson fermions), where chiral symmetry is broken at the scale of the lattice spacing. When chiral symmetry is violated, we are led to scaling corrections of order $O(a)$, while in this case, the corrections are of order $O(a^2)$.

Indeed, we avoid fermion doubling with the use of D_N, but its locality properties are not obvious in the presence of a gauge field. D_N is not an ultra-local operator and as mentioned before, we should recover locality in more general sense. We can prove the locality of D_N for sufficiently smooth gauge fields. Consequently, the Neuberger-Dirac operator seems to satisfy all the right properties of a lattice Dirac operator that describes massless quarks [1].

The action that describes this lattice regularization of QCD is the following [1]

$$S_L = a^4 \frac{N_c}{g_0^2} \sum_{x,\mu,\nu} \left(1 - \frac{1}{N_c} \operatorname{Tr} U_{\mu\nu}(x)\right) + a^4 \sum_{i=1}^{N_f} \sum_{x,y} \bar{\psi}_i(x) D_N(x,y) \psi_i(y) \tag{5.8}$$

where N_f is the number of massless fermions considered, $N_c = 3$ is the number of colors, and $U_{\mu\nu}(x)$ is the product of links $U_\mu(x)$ along an elementary plaquette that originates at x and extends in the positive $\mu - \nu$ directions.

5.3 Overlap fermions in SQCD

Now, we want to use overlap fermions in the $\mathcal{N} = 1$ version of SQCD, in order to avoid the appearance of doublers and keep the action chirally invariant when the fermion mass vanishes.

In this case, we use naive discretization for squark fields, overlap formulation for quarks and gluinos, while gluons are defined on the links of the lattice: $U_\mu(x) = e^{igaT^\alpha u_\mu^\alpha(x+a\hat{\mu}/2)}$; α is a color index in the adjoint representation [4].

The overlap operator in the fundamental representation is mentioned in the previous section (eq. (5.2)) and follows the Ginsparg-Wilson relation (eq. (5.6)), while the quark fields transform according to equation (5.7).

The introduction of the mass term in the action is performed as follows [1], [2]:

$$a^4 \sum_{x,y} \bar{\psi}(x) \left((1 - \frac{1}{2} a m_0) \mathcal{D}_{\text{ov}}(x,y) + m_0 \delta_{x,y} \right) \psi(y) \tag{5.9}$$

By rescaling the fields, we define the bare mass:

$$m = -\frac{m_0}{1 - \frac{1}{2} a m_0}. \tag{5.10}$$

In this case, we can rewrite the mass term as:

$$\bar{\psi}(1 - \frac{1}{2} a m_0) \left(\mathcal{D}_{\text{ov}} + \frac{m_0}{1 - \frac{1}{2} a m_0} \right) \psi, \tag{5.11}$$

which has the same structure as a canonical mass Dirac action:

$$\bar{\psi}'(\mathcal{D}_{\text{ov}} - m)\psi' \tag{5.12}$$

provided we make the identification:

$$-m = \frac{m_0}{1 - \frac{1}{2} a m_0}, \qquad \psi' = \sqrt{1 - \frac{1}{2} a m_0}\ \psi. \tag{5.13}$$

As mentioned before, gluinos are Majorana fermions and their spinors satisfy the identity, known as Majorana condition:

$$\bar{\lambda}^T = \mathcal{C}\lambda \quad \Rightarrow \quad \bar{\lambda} = \lambda^T \mathcal{C}^T, \tag{5.14}$$

where $\mathcal{C}$ is the charge conjugation matrix, which satisfies $(\gamma^\mu)^T\mathcal{C}^T = \mathcal{C}\gamma^\mu$, $\mathcal{C}^T = -\mathcal{C}$, $\mathcal{C}^\dagger\mathcal{C} = 1 \quad \Rightarrow \mathcal{C}\gamma^5 = (\gamma^5)^T\mathcal{C}$ and in the bases in which two gamma matrices are symmetric and two are antisymmetric, $\mathcal{C}\gamma^5 = \gamma^5\mathcal{C}$.

In this case, the overlap operator takes the form [3]:

$$\mathcal{D}_{\rm ov}^{\rm adj} = \frac{\mu}{2}(1 + V_{\rm maj}) \tag{5.15}$$

where μ is a constant with dimensions of mass that behaves as $\sim 1/a$, and $V_{\rm maj}$ is defined as:

$$V_{\rm maj} = D_W^{\rm adj}(D_W^{\rm adj\dagger} D_W^{\rm adj})^{-1/2}, \tag{5.16}$$

where

$$D_W^{\rm adj} = -\mu + \frac{1}{2}\sum_\mu \left[\gamma_\mu(\nabla_\mu^{\rm adj} + \nabla_\mu^{*\rm adj}) - \mu a^2 \nabla_\mu^{*\rm adj}\nabla_\mu^{\rm adj}\right]. \tag{5.17}$$

The covariant derivatives in the adjoint representation are

$$\nabla_\mu^{\rm adj}\psi(x)^a = \frac{1}{a}\left[U_\mu^{\rm adj}(x)^{ab}\psi^b(x+\hat{\mu}) - \psi^a(x)\right], \tag{5.18}$$

$$\nabla_\mu^{*\,\rm adj}\psi(x)^a = \frac{1}{a}\left[\psi^a(x) - U_\mu^{\rm adj\,\dagger}(x-\hat{\mu})^{ab}\psi^b(x-\hat{\mu})\right]. \tag{5.19}$$

The adjoint link variables are constructed from the fundamental ones as

$$\left[U_\mu^{\rm adj}(x)\right]^{ab} = 2\,{\rm Tr}\left(T^a\,U_\mu(x)\,T^b\,U_\mu^\dagger(x)\right), \tag{5.20}$$

where T^a are the generators of ${\rm SU}(N_c = 3)$ in the fundamental representation.

Then, the operator $V_{\rm maj}$ is unitary ($V_{\rm maj}^\dagger V_{\rm maj} = 1$) and satisfies the relations $\mathcal{C}V_{\rm maj}\mathcal{C}^{-1} = V_{\rm maj}^T$, $\gamma_5 V_{\rm maj}\gamma_5 = V_{\rm maj}^\dagger$.

The action for gluinos takes the form:

$$a^4\sum_{x,y}\bar{\lambda}(x)\mathcal{D}_{\rm ov}^{\rm adj}(x,y)\lambda(y) = -a^4\sum_{x,y}\lambda^T(x)\mathcal{C}\mathcal{D}_{\rm ov}^{\rm adj}(x,y)\lambda(y), \tag{5.21}$$

which aligns with the small-momentum behavior of the continuum theory [3].

This action is invariant under the following infinitesimal chiral transformation acting on the gluino:

$$\delta\lambda = \gamma_5\left(1 - \frac{1}{2}(1 + V_{\rm maj})\right)\lambda. \tag{5.22}$$

To find the infinitesimal tranformation of $\bar{\lambda}$, we use the Majorana condition and the properties of $V_{\rm maj}$ and $\mathcal{C}$:

$$\begin{aligned}
\delta\bar{\lambda} = -\delta\lambda^T\mathcal{C} &= -\lambda^T\left(1-\frac{1}{2}(1+V_{\rm maj}^T)\right)\gamma_5^T\mathcal{C} \\
&= \lambda^T\left(1-\frac{1}{2}(1+V_{\rm maj}^T)\right)\gamma_5^T\mathcal{C}^T \\
&= \lambda^T\left(1-\frac{1}{2}(1+V_{\rm maj}^T)\right)\mathcal{C}^T\gamma_5 \\
&= \lambda^T\left(1-\frac{1}{2}(1+\mathcal{C}V_{\rm maj}\mathcal{C}^{-1})\right)(-\mathcal{C}\gamma_5) \\
&= \bar{\lambda}\left(1-\frac{1}{2}(1+V_{\rm maj})\right)\gamma_5
\end{aligned}$$

One can prove that this transformation leaves the gluino action invariant (eq. (5.21)) as follows:

$$\begin{aligned}
\bar{\lambda}\mathcal{D}_{\rm ov}^{\rm adj}\lambda &= \bar{\lambda}\frac{\mu}{2}(1+V_{\rm maj})\lambda \\
&\to (\bar{\lambda}+\delta\bar{\lambda})\frac{\mu}{2}(1+V_{\rm maj})(\lambda+\delta\lambda) \\
&= \bar{\lambda}\frac{\mu}{2}(1+V_{\rm maj})\lambda + \delta\bar{\lambda}\frac{\mu}{2}(1+V_{\rm maj})\lambda + \bar{\lambda}\frac{\mu}{2}(1+V_{\rm maj})\delta\lambda \\
&= \bar{\lambda}\mathcal{D}_{\rm ov}^{\rm adj}\lambda + \bar{\lambda}\left(1-\frac{1}{2}(1+V_{\rm maj})\right)\gamma_5\frac{\mu}{2}(1+V_{\rm maj})\lambda \\
&+ \bar{\lambda}\frac{\mu}{2}(1+V_{\rm maj})\gamma_5\left(1-\frac{1}{2}(1+V_{\rm maj})\right)\lambda \\
&= \bar{\lambda}\mathcal{D}_{\rm ov}^{\rm adj}\lambda + \frac{\mu}{2}\bar{\lambda}\left[\gamma_5(1+V_{\rm maj}) + (1+V_{\rm maj})\gamma_5 - (1+V_{\rm maj})\gamma_5(1+V_{\rm maj})\right]\lambda \\
&= \bar{\lambda}\mathcal{D}_{\rm ov}^{\rm adj}\lambda + \frac{\mu}{2}\bar{\lambda}\left[\gamma_5 - V_{\rm maj}\gamma_5 V_{\rm maj}\right]\lambda \\
&= \bar{\lambda}\mathcal{D}_{\rm ov}^{\rm adj}\lambda
\end{aligned} \tag{5.23}$$

where the relations $\gamma_5 V_{\rm maj}\gamma_5 = V_{\rm maj}^\dagger \Rightarrow V_{\rm maj}\gamma_5 = \gamma_5 V_{\rm maj}^\dagger$, $\quad V_{\rm maj}^\dagger V_{\rm maj} = 1$ were used.

Next, we define the covariant derivatives on squark fields as [4]:

$$\mathcal{D}_\mu A_+(x) \equiv \frac{1}{a}\Big[U_\mu(x)A_+(x+a\hat{\mu}) - A_+(x)\Big] \tag{5.24}$$

$$\mathcal{D}_\mu A_+^\dagger(x) \equiv \frac{1}{a}\Big[A_+^\dagger(x+a\hat{\mu})U_\mu^\dagger(x) - A_+^\dagger(x)\Big] \tag{5.25}$$

$$\mathcal{D}_\mu A_-(x) \equiv \frac{1}{a}\Big[A_-(x+a\hat{\mu})U_\mu^\dagger(x) - A_-(x)\Big] \tag{5.26}$$

$$\mathcal{D}_\mu A_-^\dagger(x) \equiv \frac{1}{a}\Big[U_\mu(x)A_-^\dagger(x+a\hat{\mu}) - A_-^\dagger(x)\Big] \tag{5.27}$$

We note that in eq. (5.24)-(5.27), we do not use the symmetric derivative in order to avoid "scalar doubling", which would force us to introduce a Wilson term for squarks.

Taking into account all of the above, the SQCD lattice action takes the form:

$$\begin{aligned}
\mathcal{S}^L_{\rm SQCD} = \; a^4 \sum_x \Bigg\{ & \frac{N_c}{g^2}\sum_{\mu,\nu}\left(1 - \frac{1}{N_c}\,\mathrm{Tr}\,U_{\mu\nu}(x)\right) - \sum_y \mathrm{Tr}\left(\lambda^T(x)\,\mathcal{C}\,\mathcal{D}^{\rm adj}_{\rm ov}(x,y)\,\lambda(y)\right) \\
& + \mathcal{D}_\mu A_+^\dagger(x)\mathcal{D}_\mu A_+(x) + \mathcal{D}_\mu A_-(x)\mathcal{D}_\mu A_-^\dagger(x) \\
& + \sum_y \bar{\psi}(x)\left[\left(1 - \frac{am_0}{2}\right)\mathcal{D}_{\rm ov}(x,y) + m_0\delta_{x,y}\right]\psi(y) \\
& + i\sqrt{2}g\Big(A_+^\dagger(x)\,\bar{\lambda}^\alpha(x)\,T^\alpha P_+\psi(x) - \bar{\psi}(x)\,P_-\lambda^\alpha(x)\,T^\alpha A_+(x) \\
& + A_-(x)\,\bar{\lambda}^\alpha(x)\,T^\alpha P_-\psi(x) - \bar{\psi}(x)\,P_+\lambda^\alpha(x)\,T^\alpha A_-^\dagger(x)\Big) \\
& + \frac{1}{2}g^2\left(A_+^\dagger(x)\,T^\alpha A_+(x) - A_-(x)\,T^\alpha A_-^\dagger(x)\right)^2 \\
& + m^2\left(A_+^\dagger(x)\,A_+(x) + A_-(x)\,A_-^\dagger(x)\right)\Bigg\},
\end{aligned} \tag{5.28}$$

where, as before, a is the lattice spacing, $N_c = 3$ is the number of colors, $U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x)$, and a summation over flavors is understood in the last six lines of eq. (5.28) (double summation is implied in the next-to-last line).

However, the Yukawa terms of this action are not chirally invariant. To maintain exact lattice chiral symmetry and properly formulate the supersymmetric Yukawa interactions using overlap fermions, we generalize Luescher's approach [7] to a similiar problem in regular QCD and we introduce auxiliary fermion fields:

- $\chi_\lambda^\alpha(x)$: auxiliary gluino field in the adjoint representation (Majorana)
- $\chi_\psi(x)$: auxiliary quark field, with implicit flavor index, in the fundamental representation (Dirac).

These fields are non-dynamical and vanish in the continuum limit. The auxiliary field contribution to the lattice action is:

$$\mathcal{S}_{\text{aux}} = a^4 \sum_x \left(-\mu \bar{\chi}_\lambda^\alpha(x) \chi_\lambda^\alpha(x) - \frac{2}{a} \bar{\chi}_\psi(x) \chi_\psi(x) \right), \tag{5.29}$$

where a summation over color and flavor indices is implied. Also, the extended Yukawa interactions take the form:

$$\begin{aligned} \mathcal{S}_Y = a^4 \sum_x i\sqrt{2} g \Big[& A_+^\dagger(x) \big(\bar{\lambda}^\alpha(x) + \bar{\chi}_\lambda^\alpha(x)\big) T^\alpha P_+ \big(\psi(x) + \chi_\psi(x)\big) \\ & - \big(\bar{\psi}(x) + \bar{\chi}_\psi(x)\big) P_- \big(\lambda^\alpha(x) + \chi_\lambda^\alpha(x)\big) T^\alpha A_+(x) \\ & + A_-(x) \big(\bar{\lambda}^\alpha(x) + \bar{\chi}_\lambda^\alpha(x)\big) T^\alpha P_- \big(\psi(x) + \chi_\psi(x)\big) \\ & - \big(\bar{\psi}(x) + \bar{\chi}_\psi(x)\big) P_+ \big(\lambda^\alpha(x) + \chi_\lambda^\alpha(x)\big) T^\alpha A_-^\dagger(x) \Big]. \end{aligned} \tag{5.30}$$

We prove, in a similar way to eq. (5.23) and using the Ginsparg-Wilson relation (eq. (5.6)), that the new lattice action (which includes the auxiliary fields for the quark and the gluino) is invariant under the transformations given below.

$$\begin{aligned} & \delta\psi = \gamma_5 \left(1 - \frac{1}{2} a\mathcal{D}_{\text{ov}}\right) \psi + \gamma_5 \chi_\psi, && \delta\chi_\psi = \gamma_5 \frac{1}{2} a\mathcal{D}_{\text{ov}} \psi, \\ & \delta\bar{\psi} = \bar{\psi} \left(1 - \frac{1}{2} a\mathcal{D}_{\text{ov}}\right) \gamma_5 + \bar{\chi}_\psi \gamma_5, && \delta\bar{\chi}_\psi = \bar{\psi} \frac{1}{2} a\mathcal{D}_{\text{ov}} \gamma_5, \\ & \delta\lambda = \gamma_5 \left(1 - \frac{1}{2}(1 + V_{\text{maj}})\right) \lambda + \gamma_5 \chi_\lambda, && \delta\chi_\lambda = \gamma_5 \frac{1}{2}(1 + V_{\text{maj}}) \lambda, \\ \Rightarrow \quad & \delta\bar{\lambda} = \bar{\lambda} \left(1 - \frac{1}{2}(1 + V_{\text{maj}})\right) \gamma_5 + \bar{\chi}_\lambda \gamma_5, && \delta\bar{\chi}_\lambda = \bar{\lambda} \frac{1}{2}(1 + V_{\text{maj}}) \gamma_5. \end{aligned} \tag{5.31}$$

It follows from these equations that

$$\begin{aligned} & \delta(\psi + \chi_\psi) = \gamma_5(\psi + \chi_\psi), && \delta(\bar{\psi} + \bar{\chi}_\psi) = (\bar{\psi} + \bar{\chi}_\psi)\gamma_5, \\ & \delta(\lambda + \chi_\lambda) = \gamma_5(\lambda + \chi_\lambda), && \delta(\bar{\lambda} + \bar{\chi}_\lambda) = (\bar{\lambda} + \bar{\chi}_\lambda)\gamma_5. \end{aligned} \tag{5.32}$$

and the propagators of the sums $(\psi+\chi_\psi)$ and $(\lambda+\chi_\lambda)$ are, therefore, chirally invariant in the ordinary sense.

Classically, by deriving the Euler-Lagrange equations with respect to the auxiliary fields, we can see that these fields are of order $\mathcal{O}(a)$ and therefore vanish in the

continuum limit. Specifically, the Euler-Lagrange equations (differentiation of Grassmann variables is explained in Appendix A) that follow from the above action and the Majorana condition (eq. (5.14)) are:

$$-\frac{2}{a}\chi_\psi - i\sqrt{2}gT^\alpha\left(P_-A_+ + P_+A_-^\dagger\right)\chi_\lambda^\alpha = i\sqrt{2}gT^\alpha\left(P_-A_+ + P_+A_-^\dagger\right)\lambda^\alpha \quad (5.33)$$

$$\frac{2}{a}(\bar{\chi}_\psi)^T - i\sqrt{2}g(T^\alpha)^T\left(A_+^\dagger P_+ + A_-P_-\right)\mathcal{C}\chi_\lambda^\alpha = i\sqrt{2}g(T^\alpha)^T\left(A_+^\dagger P_+ + A_-P_-\right)\mathcal{C}\lambda^\alpha \quad (5.34)$$

$$\begin{aligned} 2\mu\mathcal{C}\chi_\lambda^\alpha - i\sqrt{2}g\Big[\mathcal{C}\left(A_+^\dagger P_+ + A_-P_-\right)T^\alpha\chi_\psi - \left(A_+P_- + A_-^\dagger P_+\right)(T^\alpha)^T(\bar{\chi}_\psi)^T\Big] \\ = i\sqrt{2}g\Big[\mathcal{C}\left(A_+^\dagger P_+ + A_-P_-\right)T^\alpha\psi - \left(A_+P_- + A_-^\dagger P_+\right)(T^\alpha)^T(\bar{\psi})^T\Big]. \end{aligned} \quad (5.35)$$

To reproduce the quadratic part of the action with respect to the auxiliary fields, we define the multi-spinor X

$$X \equiv \begin{pmatrix} \chi_\psi^1 \\ \vdots \\ \chi_\psi^{N_f} \\ \chi_\lambda^\alpha \end{pmatrix}, \quad (5.36)$$

where N_f is number of flavors considered in the action. In the sequel, we will also make use of the multi-spinor Ψ

$$\Psi \equiv \begin{pmatrix} -i\sqrt{2}gT^\beta\left(P_-A_+^1 + P_+A_-^{\dagger 1}\right)\lambda^\beta \\ \vdots \\ -i\sqrt{2}gT^\beta\left(P_-A_+^{N_f} + P_+A_-^{\dagger N_f}\right)\lambda^\beta \\ i\sqrt{2}g\left(A_+^\dagger P_+ + A_-P_-\right)T^\alpha\psi \end{pmatrix} \quad (5.37)$$

where a summation over flavor is understood in the last line. The components in the last line have a free color index (α) in the adjoint representation, while the other components have an implicit free color index in the fundamental representation.

We can easily write down the matrix $\mathcal{M}$ which contributes to the quadratic part of the action, $\bar{X}\mathcal{M}X$, with respect to the auxiliary fields and can be used in the functional integration of the path integral formulation. Using the definition of Ψ, the parts of the action which are linear in the auxiliary fields are expressed simply as: $\bar{X}\Psi + \bar{\Psi}X$. Starting with an action with N_f flavors, the matrix $\mathcal{M}$ takes the form

$$\mathcal{M} = \begin{pmatrix} -\frac{2}{a} & \cdots & 0 & -i\sqrt{2}gT^{\alpha}\left(P_{-}A_{+}^{1} + P_{+}A_{-}^{\dagger 1}\right) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\frac{2}{a} & -i\sqrt{2}gT^{\alpha}\left(P_{-}A_{+}^{N_f} + P_{+}A_{-}^{\dagger N_f}\right) \\ i\sqrt{2}g\left(A_{+}^{\dagger 1}P_{+} + A_{-}^{1}P_{-}\right)T^{\alpha} & \cdots & i\sqrt{2}g\left(A_{+}^{\dagger N_f}P_{+} + A_{-}^{N_f}P_{-}\right)T^{\alpha} & -\mu \end{pmatrix}$$

This is a square matrix that has $N_c \cdot N_f + (N_c^2 - 1) = 3N_f + 8$ number of rows and columns (and is also a 4×4 matrix in Dirac space).

To be able to functionally integrate the auxiliary fields out of the action, we need the determinant and the inverse matrix of $\mathcal{M}$ (see Appendix B). An important feature of the above quadratic part is that it is purely local, i.e. the matrix $\mathcal{M}$ is strictly diagonal in coordinate space. A consequence of this is that the inverse matrix appearing in the Gaussian integration over auxiliary fields is a tensor sum of the inverses of $\mathcal{M}$ at each space-time point separately. Similarly, the determinant stemming from the integration will be a product of determinants at each space-time point.

Since the projectors $P_{\pm}$ satisfy the relations $P_{+}^2 = P_{+},\ P_{-}^2 = P_{-},\ P_{+}P_{-} = P_{-}P_{+} = 0,$ and $P_{+} + P_{-} = 1$, we can write the matrix $\mathcal{M}$ as a sum of two separate matrices, $\mathcal{M}_{+}$ and $\mathcal{M}_{-}$, as follows

$$\mathcal{M} = P_{+}\mathcal{M}_{+} + P_{-}\mathcal{M}_{-} \tag{5.38}$$

where

$$\mathcal{M}_{+} = \begin{pmatrix} -\frac{2}{a} & \cdots & 0 & -i\sqrt{2}gT^{\alpha}A_{-}^{\dagger 1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\frac{2}{a} & -i\sqrt{2}gT^{\alpha}A_{-}^{\dagger N_f} \\ i\sqrt{2}gA_{+}^{\dagger 1}T^{\alpha} & \cdots & i\sqrt{2}gA_{+}^{\dagger N_f}T^{\alpha} & -\mu \end{pmatrix} \tag{5.39}$$

and

$$\mathcal{M}_{-} = \begin{pmatrix} -\frac{2}{a} & \cdots & 0 & -i\sqrt{2}gT^{\alpha}A_{+}^{1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -\frac{2}{a} & -i\sqrt{2}gT^{\alpha}A_{+}^{N_f} \\ i\sqrt{2}gA_{-}^{1}T^{\alpha} & \cdots & i\sqrt{2}gA_{-}^{N_f}T^{\alpha} & -\mu \end{pmatrix}. \tag{5.40}$$

Using the properties of the projectors, one verifies that the inverse matrix $\mathcal{M}^{-1}$ can be written as

$$\mathcal{M}^{-1} = P_{+}\mathcal{M}_{+}^{-1} + P_{-}\mathcal{M}_{-}^{-1}. \tag{5.41}$$

Every element of $\mathcal{M}_{\pm}^{-1}$ is a complicated function of the squark fields and can be

derived starting from the bottom rightmost 8×8 block, $\mathcal{N}_{\pm}$, of the corresponding inverse matrix; this block is given by:

$$\mathcal{N}_{+} = \left(\frac{a}{2} \sum_{f=1}^{N_f} \left(2g^2 A_{+}^{\dagger f} T^{\alpha} T^{\beta} A_{-}^{\dagger f} \right) - \mu \delta^{\alpha\beta} \right)^{-1}, \qquad \alpha, \beta = \{1, \ldots, 8\} \tag{5.42}$$

$$\mathcal{N}_{-} = \left(\frac{a}{2} \sum_{f=1}^{N_f} \left(2g^2 A_{-}^{f} T^{\alpha} T^{\beta} A_{+}^{f} \right) - \mu \delta^{\alpha\beta} \right)^{-1}, \qquad \alpha, \beta = \{1, \ldots, 8\}. \tag{5.43}$$

Gaussian integration will lead to a factor $\det(\mathcal{M})$ in the functional integral, as well as an additional term in the Lagrangian, $-\bar{\Psi}\mathcal{M}^{-1}\Psi$.

It can be proven that the determinant of $\mathcal{M}$ is related to the matrices $\mathcal{M}_{+}$ and $\mathcal{M}_{-}$ as follows

$$\det(\mathcal{M}) = (\det(\mathcal{M}_{+}))^2 \times (\det(\mathcal{M}_{-}))^2. \tag{5.44}$$

From the block structure of $\mathcal{M}_{\pm}$ (eqs. (5.39), (5.40)), it follows that

$$\begin{aligned} \det(\mathcal{M}_{+}) &= \left(\prod_{f=1}^{N_f} \det\left(-\frac{2}{a} \mathbb{1}_{3\times 3} \right) \right) \det\left(\frac{a}{2} \sum_{f=1}^{N_f} \left(2g^2 A_{+}^{\dagger f} T^{\alpha} T^{\beta} A_{-}^{\dagger f} \right) - \mu \delta^{\alpha\beta} \right) \\ &= \left(-\frac{2}{a} \right)^{3N_f} \det\left(\frac{a}{2} \sum_{f=1}^{N_f} \left(2g^2 A_{+}^{\dagger f} T^{\alpha} T^{\beta} A_{-}^{\dagger f} \right) - \mu \delta^{\alpha\beta} \right) \end{aligned} \tag{5.45}$$

$$\begin{aligned} \det(\mathcal{M}_{-}) &= \left(\prod_{f=1}^{N_f} \det\left(-\frac{2}{a} \mathbb{1}_{3\times 3} \right) \right) \det\left(\frac{a}{2} \sum_{f=1}^{N_f} \left(2g^2 A_{-}^{f} T^{\alpha} T^{\beta} A_{+}^{f} \right) - \mu \delta^{\alpha\beta} \right) \\ &= \left(-\frac{2}{a} \right)^{3N_f} \det\left(\frac{a}{2} \sum_{f=1}^{N_f} \left(2g^2 A_{-}^{f} T^{\alpha} T^{\beta} A_{+}^{f} \right) - \mu \delta^{\alpha\beta} \right). \end{aligned} \tag{5.46}$$

Thus the evaluation of both $\det \mathcal{M}_{+}$ and $\det \mathcal{M}_{-}$ is reduced to evaluation of determinants of 8×8 matrices.

We recall that the determinants shown in the previous equations appear at every space-time point, and a product over them is generated upon functional integration over the auxiliary fields. In a non-perturbative treatment, their evaluation thus amounts to numerical calculation at each space-time point, which is a rather straightforward task. In a perturbative treatment of the functional integral over the auxiliary fields, the constant multiplicative factors that appear in each determinant cancel out, while the part that involves squark fields will be added as an effective term to the action. We can do this using the well-known identity

$$\det(\Phi) = \exp\left(\ln\left(\det(\Phi)\right)\right) = \exp\left(\mathrm{Tr}\left(\ln\left(\Phi\right)\right)\right) \tag{5.47}$$

where Φ is an arbitrary matrix. Recalling that the functional integral will produce a product of such determinants, one for each space-time point, the term $\mathrm{Tr}\left(\ln\left(\Phi\right)\right)$ in Eq. (5.47) will be summed over all space-time, thus leading to an ultra-local contribution to the action.

Any Green function can be computed from the resulting action, which does not involve any auxiliary fields. The effect of these fields in the theory can be found in the extra terms added to the action due to the functional integration over them. This procedure establishes that the theory will be chirally invariant, while the action will not lead to any doublers.

Also, since this is an action that describes SQCD, by analogy to the continuum case, a discrete version of a gauge-fixing term and the compensating ghost term must be added to the action, to avoid divergences from the functional integration over gauge orbits. These terms are the same as in a non-supersymmetric case [4]:

$$\mathcal{S}^L_{GF} = \frac{1}{\alpha}\sum_x\sum_\mu \mathrm{Tr}\left(u_\mu(x+a\hat{\mu}/2) - u_\mu(x-a\hat{\mu}/2)\right)^2. \tag{5.48}$$

$$\begin{aligned}\mathcal{S}^L_{Ghost} =& 2a^2\sum_x\sum_\mu \mathrm{Tr}\Big((\bar{c}(x+a\hat{\mu}) - \bar{c}(x))(c(x+a\hat{\mu}) - c(x)\\ &+ ig[u_\mu(x+a\hat{\mu}/2), c(x)] + \frac{1}{2}ig[u_\mu(x+a\hat{\mu}/2), c(x+a\hat{\mu}) - c(x)]\\ &- \frac{1}{12}g^2[u_\mu(x+a\hat{\mu}/2), [u_\mu(x+a\hat{\mu}/2), c(x+a\hat{\mu}) - c(x)]]\Big) + \mathcal{O}(g^3).\end{aligned} \tag{5.49}$$

Lastly, a standard "measure" term must be added to the action, to account for the Jacobian when changing the integration variables, $U_\mu \to u_\mu$ [4]:

$$\mathcal{S}^L_M = \frac{g^2 N_c}{12}a^2\sum_x\sum_\mu \mathrm{Tr}\left(u_\mu(x+a\hat{\mu}/2)^2\right) + \mathcal{O}(g^4). \tag{5.50}$$

The chirally invariant formulation of SQCD on the lattice is thus complete, and we have all the necessary ingredients to use perturbation theory to find the quantum contributions to any Green function, and to any order of the coupling constant. This formulation allows us to renormalize the theory and fine-tune the bare parameters to the lattice spacing.

Conclusion

In this project, we studied the use of overlap fermions in a $\mathcal{N} = 1$ version of SQCD; a theory that can be a valuable tool for Standard Model investigations. According to the Nielsen-Ninomiya theorem, since the theory is now local, and not ultra-local, this approach will not lead to any doublers, while leaving the theory (for vanishing mass) chirally invariant in a modified sense. Specifically, we prepared the lattice action that can be used both in non-perturbative and in perturbative calculations of Green functions, the renormalization of the theory and the fine-tuning of the bare parameters.

A natural extension of this project would be to renormalize the theory. First, we can renormalize the fields and the masses. This will allow us to find out if the squarks will acquire a critical mass due to SUSY-breaking effects on the lattice, even though in this formulation, quarks are protected from additive renormalization. Furthermore, the renormalization will show if we have mixing of squarks and if the second Yukawa term that comes up in most lattice formulations will vanish due to chiral symmetry and therefore the supersymmetric structure is better preserved. Moreover, the renormalization of the quartic couplings will show us if there will be as many different types of such couplings as found in the Wilson formulation [4]. Lastly, in order to compute masses of bound states, one can find the composite operators (e.g. quark bilinears, the gluino-glue operator).

Appendix A

Grassmann Variables

The study of Grassmann algebra is shown here just as presented by Zinn-Justin (2021) [12].

A.1 Grassmann Algebras

Grassmann variables are introduced, as anticommuting classical variables, in order to work with fermionic Green functions which are antisymmetric with respect to the exchange of two arguments.

Grassmann Algebra

A Grassmann (or exterior) algebra $\mathfrak{A}$ over $\mathbb{R}$ or $\mathbb{C}$ (real or complex) is an associative algebra constructed from a unit 1, and a set of generators θ_i with anticommuting products:

$$\theta_i\theta_j + \theta_j\theta_i = 0, \ \forall i, j \tag{A.1}$$

Consequently:

1. If the number n of generators is finite, the elements of the algebra form a vector space of finite dimension 2^n over $\mathbb{R}$ or $\mathbb{C}$. We can write all the elements as linear combinations of the elements A_ν, $\nu = 1, ..., 2^n$ of the form

$$A_\nu \in \{1 \text{ and } \{\theta_{i_1}\theta_{i_2}...\theta_{i_p}\}, \text{with } i_1 < i_2 < ... < i_p, 1 \leq p \leq n\}. \tag{A.2}$$

2. $\mathfrak{A}$ is a graded algebra: to each monomial $\theta_{i_1}\theta_{i_2}...\theta_{i_p}$, one can associate an integer p that counts the number of generators in a product. Specifically, if A_p and A_q are monomials of degree p and q, respectively, then

$$A_pA_q = (-1)^{pq}A_qA_p.$$

3. $\mathfrak{A}$ elements are invertible if, and only if the term of degree 0 in the expansion of the basis (A.2) does not vanish.

   For instance, the element $1-\theta$ is invertible and its inverse is the element $1+\theta$, but the element θ is not invertible. The calculation of the inverse is through formal power series expansion beginning of the inverse of the term of 0th degree.

4. All elements in a Grassmann algebra, considered as functions of a generator θ_i, are first degree polynomials (namely affine functions).

Grassmannian Parity

In the exterior algebra, we can define a simple automorphism $\mathbf{P}$:

$$\mathbf{P}(\theta_i) = -\theta_i \Rightarrow \mathbf{P}^2 = \mathbf{1}. \tag{A.3}$$

Then, on a monomial of p^{th} degree, $\mathbf{P}$ acts like

$$\mathbf{P}(\theta_{i_1}...\theta_{i_p}) = (-1)^p \theta_{i_1}...\theta_{i_p}. \tag{A.4}$$

The reflection $\mathbf{P}$ separates the exterior algebra in two eigenspaces $\mathfrak{A}^{\pm}$ that contain the even or odd elements

$$\mathbf{P}(\mathfrak{A}^{\pm}) = \pm\mathfrak{A}^{\pm}$$

The algebra $\mathfrak{A}$ is a subalgebra, the subalgebra of commuting elements.

A.2 Differentiation and integration in Grassmann algebras

Differentiation in Grassmann algebras

Because of the non-commutative nature of the algebra, it is difficult to generalize simply the usual rules of differentiation. However, we can suitably define differentiation as follows: considered as functions of a specific generator θ_i, we can write all elements A of $\mathfrak{A}$

$$A = A_1 + \theta_i A_2,$$

after some commutations, where A_1 and A_2 are independent of θ_i. Then, we can define

$$\frac{\partial A}{\partial \theta_i} = A_2. \tag{A.5}$$

We can see that the square of this differential operator $\partial/\partial\theta_i$, vanishes, $(\partial/\partial\theta_i)^2 = 0$.

Left and right differentiation

Since the action of $\partial/\partial\theta_i$ consists in bringing θ_i on the left in a monomial and suppressing it, equation (A.5) is a definition of left differentiation. In the same way, we can define a right differentiation, by commuting θ_i to the right.

Chain rule

It can be verified that Grassmann differentiation also obeys the chain rule. If $\sigma(\theta)$ belongs to $\mathfrak{A}^-$ and $x(\theta)$ belongs to $\mathfrak{A}^+$, it can be found that

$$\frac{\partial}{\partial\theta}f(\sigma,x) = \frac{\partial\sigma}{\partial\theta}\frac{\partial f}{\partial\sigma} + \frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x} \tag{A.6}$$

where the order between factors of the second term, matters.

Formal construction

One can check that any Grassmann differential operator, or anti-derivation $\mathbf{D}$, acting on $\mathfrak{A}$ and defined as in equation (A.5), satisfies the two algebraic formal rules:

1. It is a linear mapping of $\mathfrak{A}$, considered as a vector space into intself:

$$\mathbf{D}(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1\mathbf{D}(A_1) + \lambda_2\mathbf{D}(A_2), \text{ for } \lambda_1,\lambda_2 \in \mathbb{R} \textit{ or } \mathbb{C}. \tag{A.7}$$

2. As opposed to the Leibinz's rule, the following condition is satisfied

$$\mathbf{D}(A_1A_2) = \mathbf{P}(A_1)\mathbf{D}(A_2) + \mathbf{D}(A_1)A_2. \tag{A.8}$$

This condition is unusual (in respect to commuting variables) but necessary, in order to satisfy an anticommuting relation between $\mathbf{D}$ and $\mathbf{P}$:

$$\mathbf{DP} + \mathbf{PD} = 0, \tag{A.9}$$

which means that the image of $\mathfrak{A}^\pm$ by $\mathbf{D}$ belongs to $\mathfrak{A}^\mp$. One can note that if A belongs to $\mathfrak{A}^+$ and $F(x)$ is an ordinary function of real or complex variables, then,

$$\mathbf{D}[F(A)] = \mathbf{D}(A)F'(A), \text{ for } A \in \mathfrak{A}^+. \tag{A.10}$$

Anticommutation relations

One can verify that if $\mathbf{D}$ and $\mathbf{D}'$ are two operators that satisfy conditions (A.7) and (A.8), then the anticommutator

$$\mathbf{\Delta} = \mathbf{D}\mathbf{D}' + \mathbf{D}'\mathbf{D} \tag{A.11}$$

is a usual differential operator:

$$\begin{aligned} &\mathbf{\Delta}(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \mathbf{\Delta}(A_1) + \lambda_2 \mathbf{\Delta}(A_2), \\ &\mathbf{\Delta}(A_1 A_2) = \mathbf{\Delta}(A_1) A_2 + A_1 \mathbf{\Delta}(A_2). \end{aligned} \tag{A.12}$$

Moreover,

$$\mathbf{\Delta}\mathbf{P} = \mathbf{P}\mathbf{\Delta} \tag{A.13}$$

These properties, are the reason we are able to extend the notion of Lie algebra, and are directly relevant to supersymmetric discussions.

Integration in Grassmann algebras

We can define, for convenience, integration over Grassmann variables. We are using the integral notation but integration and differentiation are identical operations,

$$\int \mathrm{d}\theta_i A \equiv \frac{\partial}{\partial \theta_i} A, \ \forall A \in \mathfrak{A}. \tag{A.14}$$

The integration or differentiation are used depending on the context.

General properties

We prove that Grassmann integration follows a definite integral's properties. In general, a given operator $\mathbf{D}$ is associated with an operator $\mathbf{I}$ that should have the given defining properties: it is a linear operator acting on $\mathfrak{A}$,

$$\mathbf{I}(\lambda_1 A_1 + \lambda_2 A_2) = \lambda_1 \mathbf{I}(A_1) + \lambda_2 \mathbf{I}(A_2), \tag{A.15}$$

and satisfies the three properties,

$$\mathbf{I}\mathbf{D} = 0, \tag{A.16}$$

$$\mathbf{D}\mathbf{I} = 0, \tag{A.17}$$

$$\mathbf{D}(A) = 0 \Rightarrow \mathbf{I}(BA) = \mathbf{I}(B)A. \tag{A.18}$$

Also, it changes the grading as a differential operator does:

$$\mathbf{P}\mathbf{I} + \mathbf{I}\mathbf{P} = 0.$$

The above conditions mean that: (A.16) if there are no boundary terms, the integral of a total derivative vanishes; (A.17) if we integrate over a variable, the result is now independent of the variable, and (A.18) a factor with a vanishing derivative can be taken out of the integral.

In the case of Grassmann algebras, if $\mathbf{D}^2 = 0$, $\mathbf{D}$ itself satisfies all conditions, since the differential operator's $(\partial/\partial\theta_i)$ square vanishes.

Change of variables in a Grassmann integral

We consider the integral

$$\int \mathrm{d}\theta f(\theta), \tag{A.19}$$

and change the variables (in a necessarily affine way):

$$\theta = a\theta' + b, \tag{A.20}$$

in which parity conservation implies that $a \in \mathfrak{A}^+$ and $b \in \mathfrak{A}^-$. The element a should be invertible (its term of 0th degree should be nonzero). Then, according to (A.14), we show that

$$\int \mathrm{d}\theta f(\theta) = a^{-1} \int \mathrm{d}\theta' f(\theta' a + b) = \int \mathrm{d}\theta' \left(\frac{\partial\theta}{\partial\theta'}\right)^{-1} f(\theta(\theta')), \tag{A.21}$$

where the final form does not depend on a specific parametrization. This is a very important property of Grassmann integrals: the Jacobian is a^{-1}, instead of a, which is the case for commuting variables.

Appendix B

Gaussian Integral

The study of Gaussian integrals of real, complex and Grassmann variables, and the perturbative expansion is displayed here just as described by Zinn-Justin (2021) [12].

B.1 Gaussian integrals with real variables

We consider a Gaussian integral in n-dimensions over real variables $x_i, i = 1, ..., n$, of the form,

$$\mathcal{Z}(\mathbf{S}) = \int_{\mathbb{R}^n} \mathrm{d}^n x \; e^{-S_2(\mathbf{x})}, \tag{B.1}$$

where S_2 is the real positive quadratic form,

$$S_2(\mathbf{x}) \equiv \frac{1}{2} \sum_{i,j=1}^{n} x_i S_{ij} x_j, \quad \text{with} \quad S_{ij} = S_{ji}, \quad \text{and} \quad \mathbf{S} > 0. \tag{B.2}$$

Because the matrix $\mathbf{S}$ is strictly positive, one can diagonalize it by an orthogonal transformation $\mathbf{O}$, and has positive eigenvalues s_i. By a change of variables, $x_i \mapsto x'_i$ that satisfy

$$\sum_j O_{ij} x_j = x'_i, \quad |\det \mathbf{O}| = 1,$$

a transformation of Jacobian unity, we will have a product of integrals that do not depend on x'_i.

We will obtain a factor of $\sqrt{2\pi/s_i}$ from each integral. The final outcome will involve the product of all eigenvalues, namely, the determinant. The result is

$$\mathcal{Z}(\mathbf{S}) = (2\pi)^{n/2} (\det \mathbf{S})^{-1/2}. \tag{B.3}$$

Additionally, because both the initial integral and the determinant are analytic functions of the coefficients of the matrix $\mathbf{S}$, the identity will also apply to complex matrices (with special care for the global sign).

We now focus on the more general integral

$$\mathcal{Z}(\mathbf{S},\mathbf{b}) = \int \mathrm{d}^n x \; e^{-S_2(\mathbf{x})+\mathbf{b}\cdot\mathbf{x}}, \text{ where } \mathbf{b}\cdot\mathbf{x} \equiv \sum_{i=1}^{n} b_i x_i. \tag{B.4}$$

To determine $\mathcal{Z}(\mathbf{S},\mathbf{b})$, firstly we must find the minimum of the quadratic form, given by

$$\frac{\partial}{\partial x_i}(S_2(\mathbf{x}) - \mathbf{b}\cdot\mathbf{x}) = 0 \;\Rightarrow\; \sum_j S_{ij}x_j = b_i.$$

The solution is

$$x_i = \sum_j \Delta_{ij} b_j, \text{ with } \mathbf{\Delta S} = \mathbf{1}. \tag{B.5}$$

We then change variables $\mathbf{x} \mapsto \mathbf{y}$ in the following way

$$x_i = \sum_j \Delta_{ij} b_j + y_i \;\Rightarrow\; -S_2(\mathbf{x}) + \mathbf{b}\cdot\mathbf{x} = w_2(\mathbf{b}) - S_2(\mathbf{y}), \tag{B.6}$$

with

$$w_2(\mathbf{b}) = \frac{1}{2}\sum_{i,j=1}^{n} b_i \Delta_{ij} b_j. \tag{B.7}$$

Then the final result for the integral is

$$\mathcal{Z}(\mathbf{S},\mathbf{b}) = e^{w_2(\mathbf{b})}\int \mathrm{d}^n y \; e^{-S_2(\mathbf{y})} = (2\pi)^{n/2}(\det \mathbf{S})^{(-1/2)} e^{w_2(\mathbf{b})}. \tag{B.8}$$

Gaussian expectation values

We examine the Gaussian expectation values,

$$\langle x_{k_1} x_{k_2} ... x_{k_l}\rangle \equiv \mathcal{Z}^{-1}(\mathbf{S},0)\int \mathrm{d}^n x \; x_{k_1}x_{k_2}...x_{k_l} \; e^{-S_2(\mathbf{x})}, \tag{B.9}$$

in which the normalization is determined by the condition $\langle 1\rangle = 1$.

From equation (B.4), one shows that

$$\frac{\partial}{\partial b_k}\mathcal{Z}(\mathbf{S},\mathbf{b}) = \int \mathrm{d}^n x \; x_k \; e^{-S_2(\mathbf{x}+\mathbf{b}\cdot\mathbf{x})}. \tag{B.10}$$

By differentiating repeatedly with respect to $\mathbf{b}$, we arrive at the identity

$$\langle x_{k_1} x_{k_2} ... x_{k_l} \rangle = (2\pi)^{-n/2} (\det \mathbf{S})^{1/2} \left[\frac{\partial}{\partial b_{k_1}} \frac{\partial}{\partial b_{k_2}} ... \frac{\partial}{\partial b_{k_l}} \mathcal{Z}(\mathbf{S}, \mathbf{b}) \right] \Big|_{\mathbf{b}=0}.$$

By applying the form (B.8) as the integral $\mathcal{Z}$ ($\mathbf{S}$,$\mathbf{b}$), we get the result

$$\langle x_{k_1} x_{k_2} ... x_{k_l} \rangle = \left\{ \frac{\partial}{\partial b_{k_1}} ... \frac{\partial}{\partial b_{k_l}} \; e^{w_2(\mathbf{b})} \right\} \Big|_{\mathbf{b}=0}. \tag{B.11}$$

More generally, if $F(x)$ is a power series in the variables x_i, then

$$\langle F(x) \rangle = \left[F(\partial/\partial b) \; e^{w_2(\mathbf{b})} \right] \Big|_{\mathbf{b}=0} \tag{B.12}$$

Wick's theorem

We can derive a first form of Wick's theorem from the identity (B.11). Every time that $\partial/\partial b$ acts on the exponential, a factor b is generated. The corresponding contribution will vanish when $\mathbf{b}$=0 is set, unless a second differential operator acts on the same factor. This will result in the given expression for the expectation value of the product $x_{k_1} ... x_{k_l}$ with the normalized Gaussian $e^{-S_2(\mathbf{x})}$: one must think of all the possible pairings of the indices $k_1, ..., k_l$ (so l must be even); to each pair k_p, k_q, an element $\Delta_{k_p k_q}$ of the matrix $\mathbf{\Delta}$ is associated. Subsequently,

$$\langle x_{k_1} ... x_{k_l} \rangle = \sum_{\substack{\text{all possible pairings} \\ P \text{ of } \{k_1,...,k_l\}}} \Delta_{k_{P_1} k_{P_2}} ... \Delta_{k_{P_{l-1}} k_{P_l}}, \tag{B.13}$$

$$= \sum_{\substack{\text{all possible pairings} \\ P \text{ of } \{k_1,...,k_l\}}} \langle x_{k_{P_1}} x_{k_{P_2}} \rangle ... \langle x_{k_{P_{l-1}}} x_{k_{P_l}} \rangle. \tag{B.14}$$

Wick's Theorem (equations (B.13) and (B.14)) can be generalized immediately to an infinite number of variables and, consequently, can be used in statistical and quantum theories, where functional integrals appear.

B.2 Perturbative expansion

We consider the more general integral

$$\mathcal{Z}(\lambda) = \int \mathrm{d}^n x \, \exp(-S_2(\mathbf{x}) - \lambda V(x)), \tag{B.15}$$

in which $S_2(\mathbf{x})$ is the quadratic form, $V(x)$ is a polynomial in the variables x_i and λ a parameter. In order to obtain a result for the integral, we expand the integrand in powers of λ. That is,

$$\mathcal{Z}(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int \mathrm{d}^n x \; e^{-S_2(\mathbf{x})} V^k(x).$$

The successive terms in the expansion are proportional to Gaussian expectation values of polynomials, which can be calculated using Wick's theorem:

$$\mathcal{Z}(\lambda) = \mathcal{Z}(0) \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \langle V^k(x) \rangle, \tag{B.16}$$

where $\langle \bullet \rangle$ means Gaussian expectation value. Because the function $e^{-\lambda V}$ has a power series expansion in x, we can also get a formal expression of the integral by applying the identity (B.12) with $F = e^{-\lambda V}$:

$$\mathcal{Z}(\lambda) = \mathcal{Z}(0) \left\{ \exp\left[-\lambda V\left(\frac{\partial}{\partial b} \right) \right] \right\} \Big|_{\mathbf{b}=0}. \tag{B.17}$$

B.3 Gaussian integrals with complex structures

We consider the set of $2n$ complex variables $\{z_i, \bar{z}_i\}$. The variables z_i and $\bar{z}_i$ are independent variables, and only formally conjugate, and can yield a complex parametrization of phase space, associated to the real position q_i and conjugate momentum p_i through the expressions

$$z_i = -i(p_i + iq_i)/\sqrt{2}, \qquad \bar{z}_i = i(p_i - iq_i)/\sqrt{2}. \tag{B.18}$$

We have the Gaussian integral,

$$\mathcal{Z}(\mathbf{S}) = \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2i\pi} \right) \exp\left(-\sum_{i,j=1}^{n} \bar{z}_i S_{ij} z_j \right), \tag{B.19}$$

in which $\mathbf{S}$ is a complex matrix with a non-vanishing determinant. One can compute the integral by a change of variables to the real variables (p_i, q_i), as defined in equation (B.18) (and $\mathrm{d}z_i \mathrm{d}\bar{z}_i = i\mathrm{d}p_i \mathrm{d}q_i$), or as $\sum_j S_{ij} z_j = z'_i$. Then, we obtain

$$\mathcal{Z}(\mathbf{S}) = (\det \mathbf{S})^{-1}. \tag{B.20}$$

The more general Gaussian integral is

$$\mathcal{Z}(\mathbf{S};\mathbf{b},\bar{\mathbf{b}}) = \int \left(\prod_{i=1}^{n} \frac{\mathrm{d}z_i \mathrm{d}\bar{z}_i}{2i\pi}\right) \exp\left(-\sum_{i,j=1}^{n} \bar{z}_i S_{ij} z_j + \sum_{i=1}^{n} (\bar{b}_i z_i + b_i \bar{z}_i)\right), \tag{B.21}$$

We can calculate the integral in the same way we did in the real case. We can define the complex matrix $\mathbf{\Delta} = \mathbf{S}^{-1}$ and we can eliminate any terms linear to the variables z_i and $\bar{z}_i$ with a change of variables $z_i \mapsto v_i, \bar{z}_i \mapsto \bar{v}_i$, in the following way

$$z_i = v_i + \sum_j \Delta_{ij} b_j, \quad \bar{z}_i = \bar{v}_i + \sum_j \bar{b}_j \Delta_{ji}. \tag{B.22}$$

The final expression is

$$\mathcal{Z}(\mathbf{S};\mathbf{b},\bar{\mathbf{b}}) = (\det \mathbf{S})^{-1} \exp\left(\sum_{i,j=1}^{n} \bar{b}_i \Delta_{ij} b_j\right). \tag{B.23}$$

Wick's theorem

We consider expectation values of polynomials in $z, \bar{z}$ with the normalized weight $\exp(-\sum_{i,j} \bar{z}_i S_{ij} z_j)/\mathcal{Z}(\mathbf{S})$. They can be derived by differentiation of the expression (B.23) with respect to b_i and $\bar{b}_j$, and then setting $\mathbf{b} = \bar{\mathbf{b}} = 0$.

The contribution of each derivative with respect to b will vanish when setting $\mathbf{b} = \bar{\mathbf{b}} = 0$, unless it is combined with a derivative with respect to $\bar{b}$. Only monomials with an equal number of factors z and $\bar{z}$ do not have a vanishing expectation value. So, Wick's theorem for complex integrals takes the form

$$\begin{aligned} \langle z_{i_1} \bar{z}_{j_1} \ldots z_{i_l} \bar{z}_{j_l} \rangle &= \sum_{\substack{\text{all possible pairings} \\ P \text{ of } \{j_1,\ldots,j_l\}}} \Delta_{i_1 j_{P_1}} \Delta_{i_2 j_{P_2}} \ldots \Delta_{i_l j_{P_l}}, \\ &= \sum_{\substack{\text{all possible pairings} \\ P \text{ of } \{j_1,\ldots,j_l\}}} \langle z_{i_1} \bar{z}_{j_{P_1}} \rangle \langle z_{i_2} \bar{z}_{j_{P_2}} \rangle \ldots \langle z_{i_l} \bar{z}_{j_{P_l}} \rangle. \end{aligned} \tag{B.24}$$

Conjugation

We can define a conjugation, similar to the complex conjugation, by the transformation that acts as complex conjugation on complex numbers and exchanges z_i and $\bar{z}_i$. For instance,

$$\sum_{i,j} \bar{z}_i S_{ij} z_j \mapsto \sum_{i,j} z_i S^*_{ij} \bar{z}_j.$$

A function of the $2n$ $z, \bar{z}$ variables is formally real if it is invariant under such a conjugation. For instance,

$$\sum_{i,j} \bar{z}_i S_{ij} z_j = \sum_{i,j} z_i S^*_{ij} \bar{z}_j = \sum_{i,j} \bar{z}_i S^*_{ji} z_j,$$

and, therefore, the matrix $\mathbf{S}$ us Hermitian. It can be shown that every formally real function integrated over a conjugated pair $z, \bar{z}$, again gives a formally real function.

B.4 Gaussian integrals with Grassmann variables

We consider a Grassmann algebra $\mathfrak{B}$ in which the generators are seperated into two conjugated sets. We denote by θ_i and $\bar{\theta}_i$, $i = 1, ..., n$, the respective generators.

Complex conjugation

We can define a complex conjugation which exchanges θ_i and $\bar{\theta}_i$, and it has the same properties as a Hermitian conjugation of operators: it acts by complex conjugation on complex numbers and

$$\theta_i^\dagger = \bar{\theta}_i, \ \bar{\theta}_i^\dagger = \theta_i, \ (A_1 A_2)^\dagger = A_2^\dagger A_1^\dagger, \ \forall A_1, A_2 \in \mathfrak{B}. \tag{B.25}$$

A quantity that is invariant is formally real (like the integration measure $\mathrm{d}\theta_i \mathrm{d}\bar{\theta}_i$). One can check that if, in an integral, the integrand is formally real and one integrates over a pair of conjugated variables θ_i and $\bar{\theta}_i$, the result is still formally real.

The Gaussian integral

Firstly, we have the integral

$$\mathcal{Z} = (\mathbf{S}) \int \mathrm{d}\theta_1 \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_2 \mathrm{d}\bar{\theta}_2 ... \mathrm{d}\theta_n \mathrm{d}\bar{\theta}_n \, \exp\left(\sum_{i,j=1}^{n} \bar{\theta}_i S_{ij} \theta_j \right), \tag{B.26}$$

with S_{ij} complex. The quadratic form and, therefore, the integrand is formally real if

$$\sum_{i,j=1}^{\nu} (\bar{\theta}_i S_{ij} \theta_j)^\dagger = \sum_{i,j=1}^{\nu} \bar{\theta}_j S^*_{ij} \theta_i = \sum_{i,j=1}^{\nu} \bar{\theta}_i S^\dagger_{ij} \theta_j,$$

namely if the matrix $\mathbf{S}$ is Hermitian.

According to the rules of Grassmann integration, the result is simply the coefficient of the product $\bar{\theta}_n\theta_n...\bar{\theta}_1\theta_1$ in the expansion of the integrand. We can write the integrand again as

$$\exp\left(\sum_{i,j=1}^{n}\bar{\theta}_i S_{ij}\theta_j\right) = \prod_{i=1}^{n}\exp\left(\bar{\theta}_i\sum_{j=1}^{n}S_{ij}\theta_j\right) = \prod_{i=1}^{n}\left(1+\bar{\theta}_i\sum_{j=1}^{n}S_{ij}\theta_j\right).$$

In each factor, only the term proportional to $\bar{\theta}$ will contribute to the integral. In the expansion of the product, the terms that give non-vanishing contributions to the integral are therefore of the form

$$\sum_{\substack{\text{permutations}\\ \{j_1,...,j_n\}}} S_{nj_n}S_{n-1j_{n-1}}...S_{1j_1}\bar{\theta}_n\theta_{j_n}...\bar{\theta}_1\theta_{j_1}.$$

A commutation of the generators to put them in standard order $\bar{\theta}_n\theta_n...\bar{\theta}_1\theta_1$, generates a sign, called the signature of the permutation. Then one can easily point out that the coefficient coincides with the determinant of S_{ij}. Consequently,

$$\mathcal{Z}(\mathbf{S}) = \det\mathbf{S}. \tag{B.27}$$

This result is the inverse of the one we ended up with complex commuting variables. In perturbative expansions, this will yield a sign $(-1)^L$ in front of the Feynman diagrams with L fermion loops.

General Gaussian integrals

We consider two more sets of Grassmann generators $\{\eta_i\}$ and $\{\bar{\eta}_i\}$, $i = 1, ..., n$, and the larger Grassmann algebra $\mathfrak{C}$ generated by the set of $4n$ generators $\{\theta_i, \bar{\theta}_i, \eta_i, \bar{\eta}_i\}$. Firstly we will obtain a result for the integral

$$\mathcal{Z}_G(\eta,\bar{\eta}) = \int\prod_i \mathrm{d}\theta_i\mathrm{d}\bar{\theta}_i\,\exp\left[\sum_{i,j=1}^{n}S_{ij}\bar{\theta}_i\theta_j + \sum_{i=1}^{n}(\bar{\eta}_i\theta_i + \bar{\theta}_i\eta_i)\right], \tag{B.28}$$

in which the integral is an element of $\mathfrak{C}$, and S_{ij} is a complex matrix with non-vanishing determinant.

Once again, we define $\mathbf{\Delta} = \mathbf{S}^{-1}$ and change variables $\theta \mapsto \theta'$, $\bar{\theta} \mapsto \bar{\theta}'$ as follows

$$\theta_i = \theta'_i - \sum_j \Delta_{ij}\eta_j,\ \bar{\theta}_i = \bar{\theta}'_i - \sum_j \bar{\eta}_j\Delta_{ji},$$

and we obtain the result

$$\mathcal{Z}_G(\eta,\bar{\eta}) = \det \mathbf{S}\, \exp\left(-\sum_{i,j=1}^{n} \bar{\eta}_i \Delta_{ij} \eta_j\right). \tag{B.29}$$

We can obtain, from the definition of $\mathcal{Z}_G$ and the expectation values ($\langle\bullet\rangle$) with respect to the Gaussian weight of equation (B.28), the following relations

$$\frac{\partial}{\partial\bar{\eta}_i}\mathcal{Z}_G = \det \mathbf{S}\langle\theta_i\rangle, \tag{B.30}$$

$$\frac{\partial}{\partial\eta_i}\mathcal{Z}_G = \det \mathbf{S}\langle-\bar{\theta}_i\rangle \tag{B.31}$$

Wick's theorem for Grassmann integrals

The definition of Gaussian expectation values is

$$\begin{aligned}&\langle\bar{\theta}_{i_1}\theta_{j_1}\bar{\theta}_{i_2}\theta_{j_2}...\bar{\theta}_{i_n}\theta_{j_n}\rangle \\ &= \frac{1}{\det \mathbf{S}}\int\left(\prod_i \mathrm{d}\theta_i \mathrm{d}\bar{\theta}_i\right)\bar{\theta}_{i_1}\theta_{j_1}...\bar{\theta}_{i_n}\theta_{j_n}\; exp\left(\sum_{i,j=1}^{n}\bar{\theta}_i S_{ij}\theta_j\right),\end{aligned} \tag{B.32}$$

From equations (B.30) and (B.31), it follows that

$$\det \mathbf{S}\langle\bar{\theta}_{i_1}\theta_{j_1}\bar{\theta}_{i_2}\theta_{j_2}...\bar{\theta}_{i_n}\theta_{j_n}\rangle = \left[\frac{\partial}{\partial\bar{\eta}_{j_1}}\frac{\partial}{\partial\eta_{i_1}}...\frac{\partial}{\partial\bar{\eta}_{j_n}}\frac{\partial}{\partial\eta_{j_n}}\mathcal{Z}_G(\eta,\bar{\eta})\right]\Big|_{\eta=\bar{\eta}=0} \tag{B.33}$$

and after applying the expression (B.29),

$$\begin{aligned}&\langle\bar{\theta}_{i_1}\theta_{j_1}\bar{\theta}_{i_2}\theta_{j_2}...\bar{\theta}_{i_n}\theta_{j_n}\rangle \\ &= \left\{\frac{\partial}{\partial\bar{\eta}_{j_1}}\frac{\partial}{\partial\eta_{i_1}}...\frac{\partial}{\partial\bar{\eta}_{j_n}}\frac{\partial}{\partial\eta_{j_n}}\; exp\left[-\sum_{i,j=1}^{n}\bar{\eta}_j\Delta_{ji}\eta_i\right]\right\}\Big|_{\eta=\bar{\eta}=0}.\end{aligned} \tag{B.34}$$

By explicit differentiation (which is identical to integration), we can get arrive to the result

$$\begin{aligned}\langle\bar{\theta}_{i_1}\theta_{j_1}...\bar{\theta}_{i_n}\theta_{j_n}\rangle &= \det\Delta_{j_l i_k} = \det\langle\bar{\theta}_{i_k}\theta_{j_l}\rangle \\ &= \sum_{\substack{\text{permutations } P \\ of\ \{j_1,...,j_n\}}} \mathrm{sgn}(P)\Delta_{j_{P_1}i_1}\Delta_{j_{P_2}i_2}...\Delta_{j_{P_n}i_n},\end{aligned} \tag{B.35}$$

where $\mathrm{sgn}(P)$ is the signature of the permutation P. This is Wick's theorem for Grassmann algebras. The result is different from the one we get when having complex commuting variables, only by the factor $\mathrm{sgn}(P)$.

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